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MODULATIONAL APPROACH TO STABILITY OF NON-TOPOLOGICAL SOLITONS IN SEMILINEAR WAVE EQUATIONS

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ABSTRACT. – Stability properties of a class of solitary wave solutions of the equation $\Box\phi + m^2\phi = \beta(|\phi|)\phi$, where $\phi:\mathbb{R}^{1+n} \to \mathbb{C}$, are studied. The solitary waves, of the form $e^{i\omega t} f(x)$, are called non-topological solitons. A modulational approach to stability is developed along the lines of that for the nonlinear Schrödinger equation; the novel features of the present analysis arise from the ω-dependence of the stability condition. This is explained at the linear level as a condition for positivity of the Hessian on the subspace *symplectically* orthogonal to the tangent space. In the case $\beta(|\phi|) = |\phi|^{p-1}$ the stability interval for ω can be determined precisely from the Gagliardo–Nirenberg inequality. The main theorem provides a strenthening, for this equation, of existing very general stability results for solitary waves in Hamiltonian systems proved by Grillakis, Shatah and Strauss. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – On étudie la stabilité des ondes solitaires, de la forme $e^{i\omega t} f_\omega(x)$, de l'équation $\Box\phi + m^2\phi = \beta(|\phi|)\phi$ où $\phi:\mathbb{R}^{1+n} \to \mathbb{C}$. On prouve la stabilité au sens modulationel ; les éléments nouveaux dans notre analyse émanent de la dépendance en ω du critére de stabilité. On explique cette dépendance en ω, au niveau linéaire, comme une condition suffisante pour la positivité de la Hessienne restreinte au sous-espace orthogonal, au sens symplectique, à l'espace tangent. Dans le cas où $\beta(|\phi|) = |\phi|^{p-1}$, on peut décrire précisément l'intervalle de stabilité grâce à l'inégalité de Gagliardo–Nirenberg. Le théorème principal constitue une amélioration, pour cette équation, des résultats généraux de stabilité pour les systèmes Hamiltoniens obtenus dans par Grillakis, Shatah et Strauss. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

1.1. Non-topological solitons

This article is devoted to a study of the stability of solitons, or solitary waves, in the equation:

$$\partial_t^2\phi - \Delta\phi + m^2\phi = \mathcal{F}(\phi), \quad \mathcal{F}(\phi) = \beta\big(|\phi|\big)\phi. \tag{1.1}$$

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Here m is a fixed positive number and β is a function for which various hypotheses will be made; an example to keep in mind is $\beta(|f|) = |f|^{p-1}$ (this will be referred to as the pure power case). More precisely this article is concerned with a class of travelling wave solutions called *non-topological solitons*, which may be written in the following way. First of all under certain conditions on β and ω, which are detailed and referenced in Section 1.3.1, there exists a unique positive, radially symmetric and decreasing solution, f_ω, in $H^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ to

$$-\Delta f_\omega + \left(m^2 - \omega^2\right) f_\omega = \mathcal{F}(f_\omega), \quad \mathcal{F}(f_\omega) = \beta(f_\omega) f_\omega. \tag{1.2}$$

The basic soliton is given by $e^{i\omega t} f_\omega(x)$; it is straightforward to see that this is a solution of (1.1). Stability (and instability) results for these solitons were obtained in [12] in the case of radial symmetry. These results were later put into a very general framework providing stability theorems for solitary waves in Hamiltonian systems in [4,5]. The present article strengthens these results for the particular case of (1.1).

The equation (1.1) may be written in first order form a

$$\begin{aligned} &\partial_t \phi = \psi, \\ &\partial \psi = \Delta\phi - m^2\phi + \beta\left(|\phi|\right)\phi. \end{aligned} \tag{1.3}$$

As discussed in Section 1.2.1 the system (1.3) is a Hamiltonian system with Hamiltonian:

$$H(\phi, \psi) = \frac{1}{2} \int\limits_{\mathbb{R}^n} \left(|\psi|^2 + |\nabla\phi|^2 + m^2|\phi|^2 - \mathcal{V}(\phi)\right) dx, \tag{1.4}$$

where

$$\mathcal{V}(\phi) = G\left(|\phi|\right), \quad \text{where } G(f) = \int\limits_0^f 2t\beta(t)\, dt.$$

The action of the Poincaré group on the basic soliton $e^{i\omega t} f_\omega(x)$ gives a $2n+2$ parameter family of solutions as follows. Let $\lambda = (\omega, \theta, \xi, u) \in O$ where $O \subset \mathbb{R}^{2n+2}$ is given by

$$O \equiv \left\{(\omega, \theta, \xi, u) \in \mathbb{R}^{2n+2}\colon |u| < 1 \text{ and } |\omega| < m\right\}. \tag{1.5}$$

Define:

$$\mathbf{Z}(x; \lambda) = \gamma P_u(x - \xi) + Q_u(x - \xi), \tag{1.6}$$

$$\boldsymbol{\Theta}(x; \lambda) = \theta - \omega u \cdot \mathbf{Z}(x; \lambda), \tag{1.7}$$

$$\gamma = \overline{\boldsymbol{\gamma}}(u) \quad \text{with } \overline{\boldsymbol{\gamma}}(u) \equiv \left(1 - |u|^2\right)^{-1/2}, \tag{1.8}$$

where $P_u : \mathbb{R}^n \to \mathbb{R}^n$ is the projection operator in the direction of $u \in \mathbb{R}^n$, $Q_u + P_u = 1$ (see (A.14)), and

$$\phi_S(x; \lambda) = e^{i\boldsymbol{\Theta}(x;\lambda)} f_\omega\left(\mathbf{Z}(x; \lambda)\right), \tag{1.9}$$

$$\psi_S(x; \lambda) = e^{i\boldsymbol{\Theta}(x;\lambda)}\left(i\omega\gamma f_\omega\left(\mathbf{Z}(x; \lambda)\right) - \gamma u \cdot \nabla_Z f_\omega\left(\mathbf{Z}(x; \lambda)\right)\right). \tag{1.10}$$

Notice the function $\lambda \to \left(\phi_S(\cdot, \lambda), \psi_S(\cdot, \lambda)\right) \in H^1 \oplus L^2$ is C^1 on O (see Section 1.3.1); the space of solitons $\mathcal{S}$ is the image of this map:

$$\mathcal{S} = \left(\phi_S(\cdot, \lambda), \psi_S(\cdot, \lambda)\right) \subset X \equiv H^1 \oplus L^2.$$

If $\lambda \in C^1(I; O)$, for some interval $I \subset \mathbb{R}$, it follows that the pair

$$\big(\phi(t,x), \psi(t,x)\big) = \big(\phi_S\big(x; \lambda(t)\big), \psi_S\big(x; \lambda(t)\big)\big)$$

satisfies (1.3) for $t \in I$ as long as $\lambda(t) = ((\omega(t), \theta(t), \xi(t), u(t))$ evolves according to

$$\begin{aligned} &\frac{\mathrm{d}u}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}\omega}{\mathrm{d}t} = 0,\\ &\frac{\mathrm{d}\xi}{\mathrm{d}t} = u, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\omega}{\gamma}. \end{aligned} \tag{1.11}$$

Introduce the corresponding vector field on O:

$$V(\lambda) = V(\omega, \theta, \xi, u) = \big(0, \omega/\overline{\boldsymbol{\gamma}}(u), u, 0\big) \tag{1.12}$$

and notice that $\psi_S = D_\lambda \phi_S(V(\lambda))$, as expected.

1.2. Stability in the pure power case and methodology

For clarity, stability will first be discussed in the context of the special case $\beta(|f|) = |f|^{p-1}$, and a discussion of the methodology given, before going onto the general case.

THEOREM 1.1. – *Consider the Cauchy problem for* (1.3), *with* $\beta(|\phi|) = |\phi|^{p-1}$, *and with initial values* $(\phi(0,\cdot), \psi(0,\cdot)) \in H^1 \oplus L^2$. *Define*:

$$\begin{aligned} O_{\mathrm{stab}} &\equiv \left\{(\omega, \theta, \xi, u) \in O\colon \frac{1}{1 + \frac{4}{p-1} - n} < \frac{\omega^2}{m^2} < 1\right\}, \quad 1 < p < 1 + \frac{4}{n},\\ &\equiv \emptyset \quad \textit{otherwise.} \end{aligned} \tag{1.13}$$

Then for all $\lambda_0 \in O_{\mathrm{stab}}$ *there exists* $\varepsilon_* = \varepsilon_*(\lambda_0) > 0$ *such that if*

$$\varepsilon = \left\|\phi(0,\cdot) - \phi_S(\cdot;\lambda_0)\right\|_{H^1} + \left\|\psi(0,\cdot) - \psi_S(\cdot;\lambda_0)\right\|_{L^2} < \varepsilon_*(\lambda_0);$$

then there exists $\lambda \in C^1(\mathbb{R}; O_{\mathrm{stab}})$, $(\phi, \psi) \in C(\mathbb{R}; H^1 \oplus L^2)$ *and* $c_1 > 0$ *such that*:

$$\sup_{t\in\mathbb{R}}\big(\left\|\phi(t,\cdot) - \phi_S\big(\cdot;\lambda(t)\big)\right\|_{H^1} + \left\|\psi(t,\cdot) - \psi_S\big(\cdot;\lambda(t)\big)\right\|_{L^2}\big) < c_1\varepsilon. \tag{1.14}$$

The curve $t \to \lambda(t)$ *is the solution of a system of ordinary differential equations* (2.46)–(2.49) *and there exists* $c_2 > 0$ *such that*:

$$\left|\partial_t \lambda - V(\lambda)\right| \leqslant c_2 \varepsilon. \tag{1.15}$$

Remarks. – This theorem is a consequence of the more general Theorem 1.6 which is proved in Section 2. The condition on ω in the definition of O_{stab} appeared in the article [12], (see [14] for further references) where it was proved in the radially symmetric situation to be a sufficient condition for stability[1]; stability was there taken in a slightly weaker sense namely that, for all t, there exists a number $\lambda(t)$ such that (1.14) holds. Thus this paper provides:

[1] It was also shown that if either $p > 1 + 4/n$ or ω^2 is less than the critical value in (1.13) the soliton is unstable.

(i) a proof of a slightly stronger statement, namely that there exists an *explicitly determined* C^1 function $t \to \lambda(t)$ such that (1.14) holds,

(ii) a proof that stability holds for the range of frequencies in (1.13), *for all velocities* u, and *without the assumption of radial symmetry*.

This velocity independence of the stability condition is certainly to be expected in view of Lorentz invariance, but does require proof since the Lorentz transform does not preserve $t =$ const. hyperplanes. Stability in the absence of radial symmetry is certainly accessible in principle by the very general theorems of [5], but does not appear to have been treated in the literature previously. The present technique differs from that in [5] in that the velocity and frequency u, ω are allowed to vary in time as well as the phase and centre θ, ξ; as explained in Section 1.2.1 this corresponds to a "symplectification" of the standard approach.

The main point of the *modulational* approach to stability developed here, however, is to prove that $\lambda(t)$ is continuously differentiable, and to derive a set of ordinary differential equations from which $\lambda(t)$ can be determined; this is carried out in Section 2.5 following the ideas outlined in Sections 1.2.1–1.2.2. This development is similar to that in [17], but there is an added complication here due to the frequency dependence of the stability condition, which is the underlying reason why the linear analysis is here more complicated. The modulation theory can lead to interesting phenomena in more complicated situations, such as the study of (1.1) on a pseudo-Riemannian manifold; in this setting it motivates a proof that there exist solutions with solitons concentrated along time-like geodesics in a certain scaling limit ([15]).

The frequency dependence in (1.13) is a noteworthy feature. In the following two sections two situations in which the condition (1.13) appears are explained: firstly, in the restriction of the symplectic form to the space of solutions, and secondly in the linearisation.

1.2.1. Hamiltonian structure

It is helpful to regard the system (1.3) as (at least formally) a Hamiltonian evolution in the vector space $X \equiv (\phi, \psi) \in H^1 \times L^2$ endowed with the symplectic structure:

$$\begin{gathered} \underline{\boldsymbol{\Omega}} : X \times X \to \mathbb{R} \\ \underline{\boldsymbol{\Omega}}\big((\phi', \psi'), (\dot\phi, \dot\psi)\big) = \int_{\mathbb{R}^n} \big(\langle \phi', \dot\psi\rangle - \langle \psi', \dot\phi\rangle\big)\, \mathrm{d}x, \end{gathered} \tag{1.16}$$

where $\langle a, b\rangle = \frac{1}{2}(\bar a b + a\bar b)$. The following simple theorem gives a first intimation of the significance of the condition (1.13):

THEOREM 1.2. – *The subset $\mathcal{S}_{\mathrm{stab}} = (\phi_S(\cdot,\lambda), \psi_S(\cdot,\lambda))|_{\lambda\in O_{\mathrm{stab}}} \subset X$ is a local C^1 symplectic submanifold, in particular $\bar{\boldsymbol{\Omega}}$, the restriction of $\underline{\boldsymbol{\Omega}}$, is non-degenerate. As λ approaches the boundary of O_{stab} in O, i.e., as ω^2/m^2 tends to $1/(1 + \frac{4}{p-1} - n)$, $\bar{\boldsymbol{\Omega}}$ degenerates.*

This is proved in Section 2.3, as a special case of Theorem 2.5.

Assume that $(\phi(t,\cdot), \psi(t,\cdot))$ is a solution to (1.3) initially close to some soliton. Then to prove Theorem 1.1 it is necessary to pick out, at each t, the soliton $(\phi_s(\cdot;\lambda(t)), \psi_s(\cdot;\lambda(t)))$ which "best approximates" $(\phi(t,\cdot), \psi(t,\cdot))$ in an appropriate sense. As usual in modulation theory this is done by requiring that $(\phi(t,\cdot) - \phi_s(\cdot;\lambda(t)), \psi(t,\cdot) - \psi_s(\cdot;\lambda(t)))$ lies in some subspace; in the present situation this subspace is $N_\lambda\mathcal{S}$ the *symplectic* normal subspace to the space of solitons $\mathcal{S}$, i.e.

$$N_\lambda\mathcal{S} = \big\{(\tilde\phi, \tilde\psi) \in X\colon \underline{\boldsymbol{\Omega}}\big((\tilde\phi, \tilde\psi), \tau_A\big) = 0\big\}, \tag{1.17}$$

where

$$\tau_A(\lambda) = \frac{\partial}{\partial\lambda_A}(\phi_s, \psi_s), \quad A = -1, \ldots, 2n+1, \tag{1.18}$$

are a basis for the tangent space $T_\lambda \mathcal{S}$. (Here the arguments of ϕ, ψ are t, x and of ϕ_s, ψ_s are $x, \lambda(t)$.) It is proved in Section 2.5 that this condition is equivalent to the requirement that $\lambda(t)$ satisfy a system of ordinary differential equations, (2.46)–(2.49), which is locally well-posed (at least if ω satisfies the stability condition in (1.13).)[2]

The basic estimates used to prove stability derive from conservation laws; as well as the energy (1.4), there are conservation laws corresponding to translational and S^1-phase symmetry:

$$\Pi_i(\phi,\psi) = \int \left\langle \psi, \frac{\partial \phi}{\partial x^i} \right\rangle \mathrm{d}x \quad \text{(momentum)}, \tag{1.19}$$

$$Q(\phi,\psi) = \int \langle \mathrm{i}\psi, \phi\rangle \,\mathrm{d}x \quad \text{(charge)}. \tag{1.20}$$

It is useful to note, at the formal level, that (ϕ_s, ψ_s) are critical points of H subject to the constraints that Π, Q be fixed, and u^i and ω/γ are the corresponding Lagrange multipliers. (Here γ is as in (1.8).) Thus ϕ_s, ψ_s are critical points of the enlarged functional

$$F(\phi,\psi;\lambda) = H(\phi,\psi) + u^i\big(\Pi_i(\phi,\psi) - p_i\big) + \frac{\omega}{\gamma}\big(Q(\phi,\psi) - q\big), \tag{1.21}$$

for appropriate p, q. Thus $\operatorname{Hess} F_\lambda$, the Hessian of F at $(\phi_S(\cdot,\lambda), \psi_S(\cdot,\lambda))$, should be an important quantity for the stability analysis, as is now discussed.

1.2.2. Linearization

The frequency dependence in (1.13) is peculiar in view of the fact that the solitons for different frequencies are all related to one another by the following rescaling:

$$f_\omega(x) = \big(m^2 - \omega^2\big)^{1/(p-1)} f\big(\sqrt{m^2-\omega^2}\,x\big), \tag{1.22}$$

where f is the corresponding solution of $-\Delta f + f = f^p$. There are two important Schrödinger operators, L_+ and L_-, which appear in the linearisation in, respectively, the real and the imaginary direction. These are unbounded operators acting on real-valued functions $v = v(Z) \in H^1(\mathbb{R}^n)$ and are given explicitly by:

$$\begin{aligned} L_+ &= -\Delta_Z + \big(m^2-\omega^2\big) - \beta\big(f_\omega(Z)\big) - f_\omega(Z)\beta'\big(f_\omega(Z)\big) \\ &= -\Delta_Z + \big(m^2-\omega^2\big) - p f_\omega^{p-1}(Z), \quad \text{if } \beta(f) = |f|^{p-1}, \end{aligned} \tag{1.23}$$

$$\begin{aligned} L_- &= -\Delta_Z + \big(m^2-\omega^2\big) - \beta\big(f_\omega(Z)\big) \\ &= -\Delta_Z + \big(m^2-\omega^2\big) - f_\omega^{p-1}(Z), \quad \text{if } \beta(f) = |f|^{p-1}. \end{aligned} \tag{1.24}$$

The scaling in (1.22) converts these also into a standard form with $m^2 - \omega^2 = 1$ and f_ω replaced by f. Thus it is an interesting question to understand *at the linear level* how the frequency dependence arises, if possible. An answer to this is given by the next theorem which follows some definitions. Introduce the pair:

$$(v,w) = \mathrm{e}^{-\mathrm{i}\Theta(\cdot;\lambda(t))}\big(\phi(t,\cdot) - \phi_s\big(\cdot;\lambda(t)\big), \psi(t,\cdot) - \psi_s\big(\cdot;\lambda(t)\big)\big)$$

and define, for each $\lambda \in O$, the quadratic form $\mathcal{E}$ on $H^1 \times L^2$ by:

$$\mathcal{E}(v,w;\lambda) = \operatorname{Hess} F_\lambda(\phi - \phi_s, \psi - \psi_s),$$

[2] For the use of such conditions to describe dynamics in a neighbourhood of a relative equilibrium in finite-dimensional Hamiltonian systems see [8] and references therein.

where Hess F_λ is the Hessian of F at (ϕ_S, ψ_S). Explicitly $\mathcal{E}$ is given by:

$$\mathcal{E}(v, w; \lambda) = \frac{1}{2}|w + \gamma u \cdot \nabla_Z v - \mathrm{i}\omega\gamma v|^2_{L^2} + \frac{1}{2}\langle v_1, L_+ v_1\rangle_{L^2} + \frac{1}{2}\langle v_2, L_- v_2\rangle_{L^2}, \tag{1.25}$$

for $v = v(Z) = v_1 + \mathrm{i}v_2 \in H^1(\mathbb{R}^n; \mathbb{C})$ and $w = w(Z) \in L^2(\mathbb{R}^n; \mathbb{C})$. Define the subspace Υ_λ by the condition that

$$(v, w) \in \Upsilon_\lambda \text{ if and only if } \big(\phi - \phi_s(\cdot; \lambda), \psi - \psi_s(\cdot; \lambda)\big) \in N_\lambda \mathcal{S}$$

(see equations (2.7)–(2.8) for an explicit description).

THEOREM 1.3. – *Assume ω is such that $\lambda = (\omega, \theta, \xi, u) \in O_{\mathrm{stab}}$, as defined in* (1.13). *Then for (v, w) restricted to lie in Υ_λ the quadratic form $\mathcal{E}(v, w; \lambda)$ is equivalent to the $H^1 \oplus L^2$ norm, uniformly for λ in compact subsets of O_{stab}.*

This theorem gives a linear interpretation of the stable frequency range which arises because the appropriate subspace on which to work is defined by the symplectic orthogonality conditions (1.17). It is proved in this case using the fact that the f_ω are optimisers of the Gagliardo–Nirenberg inequality ([16]). A more general theorem along these lines is given in Section 2.6.

1.3. General stability theorem

In order to state a stability theorem valid for more general nonlinearities it is first necessary to discuss situations under which the functions f_ω exist and are unique.

1.3.1. Existence and uniqueness of non-topological solitons

This section is concerned with summarising known existence and uniqueness results for positive, radially symmetric and decreasing functions in

$$H^s_{\mathrm{rad}}\big(\mathbb{R}^n\big) = \big\{f \in H^s\big(\mathbb{R}^n\big): \ f = f\big(|x|\big)\big\}$$

which satisfy (1.2). Since the profile function f_ω is real it is only the restriction of $\mathcal{F}$ to the real axis which is important for the properties of f_ω, and it is this function on which hypotheses will now be introduced; notice that the form assumed for $\mathcal{F}$ in (1.1) ensures that $\mathcal{F}: \mathbb{R} \to \mathbb{R}$. The following hypotheses on $\mathcal{F}$ are designed to ensure the existence and regularity of f_ω necessary in this article:

$$\mathcal{F}(f) = -\mathcal{F}(-f) \quad \text{and} \quad \mathcal{F} \in C^1(\mathbb{R}) \cap C^2((0, \infty)), \tag{E1}$$

$$\mathcal{F}(0) = \mathcal{F}'(0) = 0 \quad \text{and} \quad \lim_{f \to 0} f^s \mathcal{F}''(f) = 0 \text{ for some } s \in (0, 1), \tag{E2}$$

$$\exists \zeta > 0 \text{ such that } \mathcal{V}(\zeta) - (m^2 - \omega^2)\zeta^2 > 0, \tag{E3}$$

$$\lim_{f \to \infty} (\mathcal{F}(f)/f^l) = 0 \ \begin{cases} \text{if } l = 1 + 4/(n-2) \text{ for } n > 2 \\ \text{for some } l \in (0, +\infty) \text{ if } n = 2 \\ \text{no such growth condition if } n = 1. \end{cases} \tag{E4}$$

The next two conditions, from [10], are to ensure uniqueness:

$$\text{(U1)}\qquad \begin{aligned} &\exists\alpha > 0 \text{ such that:}\\ &\big(m^2-\omega^2\big)f - \mathcal{F}(f) > 0 \quad \text{for } 0 < f < \alpha,\\ &\big(m^2-\omega^2\big)f - \mathcal{F}(f) < 0 \quad \text{for } \alpha < f < \infty\\ &\text{and } \big(\mathcal{F}(f) - \big(m^2-\omega^2\big)f\big)' > 0 \quad \text{when } f = \alpha. \end{aligned}$$

$$\text{(U2)}\qquad \begin{aligned} &\text{For } U > \alpha\, \exists\lambda = \lambda(U) \in C\big((\alpha,\infty),\mathbb{R}^+\big) \quad \text{such that}\\ &I(f,\lambda) \geqslant 0 \quad \text{on } (0,U),\\ &I(f,\lambda) \leqslant 0 \quad \text{on } (U,\infty),\\ &\text{where} \quad I(f,\lambda) = 2\big(m^2-\omega^2\big)f + \lambda f\mathcal{F}'(f) - (\lambda+2)\mathcal{F}(f). \end{aligned}$$

Finally the following spectral assumption will also be needed:

(S1) The subspace on which L_+ is strictly negative is one-dimensional.

Remark. – In the pure power case (S1) is a consequence of the characterization of the f_ω as optimisers for the Gagliardo–Nirenberg inequality. More generally when the f_ω are obtained by constrained minimisation as in [1] it can be deduced from the necessary conditions for minimality given by the Lagrange multiplier theorems.

The next theorem summarises the relevant facts about positive solutions of (1.2).

THEOREM 1.4 [1,2,13,11,10,17]. – *Let $m^2-\omega^2 > 0$. Under the hypotheses* (E1)–(E4) *there exists a positive function $f_\omega \in H^4_{\mathrm{rad}}(\mathbb{R}^n) \cap C^4(\mathbb{R}^n)$ which satisfies* (1.2). *It is a decreasing function of $|x| \in (0,+\infty)$ which decays exponentially fast as $|x| \to +\infty$, as do its first, second and third derivatives, in the sense that*:

$$\sum_{|\alpha|\leqslant 3} \limsup_{|x|\to\infty} \nabla^\alpha f_\omega \mathrm{e}^{|x|(\sqrt{m^2-\omega^2}-\varepsilon)} < +\infty \quad \forall\varepsilon \in \big(0,\sqrt{m^2-\omega^2}\big). \tag{1.26}$$

Also

$$\lim_{|x|\to\infty} \frac{f'_\omega(|x|)}{f_\omega(|x|)} = -\sqrt{m^2-\omega^2} \tag{1.27}$$

and for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$f_\omega\big(|x|\big) \geqslant C(\varepsilon)\mathrm{e}^{-(\sqrt{m^2-\omega^2}+\varepsilon)|x|}.$$

Furthermore, f_ω is the unique such solution if either $n = 1$ or under the additional hypotheses (U1)–(U2) *for $n \geqslant 2$. In this case if L_+, L_- are the linear operators defined in* (1.23)–(1.24) *then*:

$$\operatorname{Ker} L_+ = \{\nabla_i f_\omega\}_{i=1}^n, \tag{1.28}$$

$$\operatorname{Ker} L_- = \{f_\omega\}, \tag{1.29}$$

where $\{\,\cdot\,\}$ means linear span. The map $\omega \to f_\omega \in H^k_{\mathrm{rad}}(\mathbb{R}^n)$ is continuous for $k = 4$, C^1 for $k = 3$ and C^2 for $k = 2$ (for $\omega \in (-m,+m)$), and $g_\omega = \partial_\omega f_\omega$ decays exponentially as in (1.26).

Proof. – This theorem is essentially a collection of known results, for which references will be given, together with some minor amplifications. For the existence part of Theorem 1.4 see [1,2,13] and previous references therein. The C^4 regularity assertion follows in a straightforward way from the arguments in [1, Section 4.1 and 4.2] where it is proved that f_ω is C^2, using the additional assumption (E2) on $\mathcal{F}$: for $|x|>0$ observe that although $\mathcal{F}$ is not necessarily C^2 everywhere it is C^2 on the range of f_ω by positivity and thus f_ω is easily seen to be C^4 away from the origin. To show that the derivatives up to fourth order have limits as $r=|x|\to 0$ define, following [1, Section 4.1 and 4.2], $v(r)=-(m^2-\omega^2)f_\omega(r)+\mathcal{F}(f_\omega(r))$ so that as in this reference

$$-\partial_r^2 f_\omega-\frac{n-1}{r}\partial_r f_\omega=v(r)$$

and

$$\frac{\partial_r f_\omega}{r}=-\int_0^1\tau^{n-1}v(r\tau)\,\mathrm{d}\tau.$$

In the disc $r<1$ f_ω is bounded away from zero so that $\mathcal{F}(f_\omega(r))$ is C^2, and so two derivatives of $\frac{\partial_r f_\omega}{r}$ have limits as $r\to 0$ and hence by the equation the fourth derivative of f_ω has a limit as $r\to 0$.

The assertion (1.27) is proved in [11]. To prove the Sobolev regularity statement is straightforward for H^3 since $\mathcal{F}\in C^1(\mathbb{R})$ and if the equation is differentiated once only $\mathcal{F}'$ appears. However some care is needed for H^4: the second derivative of (1.2) gives schematically (dropping the subscript ω for the moment):

$$\bigl(-\Delta+\bigl(m^2-\omega^2\bigr)\bigr)\nabla^2 f=\mathcal{F}''(f)(\nabla f)^2+\mathcal{F}'(f)\bigl(\nabla^2 f\bigr).$$

The second term on the right is clearly in L^2 (since f is already known to be bounded in H^2) and so presents no difficulty; to deal with the first term write it as:

$$f^s\mathcal{F}''(f)\frac{(\nabla f)^2}{f^s}.$$

Applying the final condition in (E2) and the upper and lower bounds for f and its derivatives it follows that this term is continuous and *exponentially decaying* (since $s<2$), and so the standard estimate gives $f\in H^4$. Notice that this treatment of the $\mathcal{F}''$ term also provides a proof that the fourth derivative of f_ω decays exponentially, although not necessarily at the same rate as in (1.26).

Uniqueness is proved for $n=1$ in [1] and for $n>1$ in [10] (using ideas of Coffman, Kwong, Zhang [11] and others referenced there). The information on the null spaces in (1.28)–(1.29) is proved in [17]. (Cases not covered in [17] can be treated identically using the fact, stated and proved in [10], that the kernel of L_+ in the radial sector is empty under assumptions (U1)–(U2).)

Uniqueness implies the stated continuity properties directly using compactness and in fact since (1.2) reduces, in the spherically symmetric case, to a single second order ordinary differential equation, the map $\omega\to f_\omega$ is in fact continuous into C^4. The proof of differentiability properties of the map $\omega\to f_\omega\in H^2$ is carried out as follows. Firstly, if $g_\omega=\partial_\omega f_\omega$ exists as stated, it is the solution of:

$$L_+g_\omega=-\Delta g_\omega+\bigl(m^2-\omega^2\bigr)g_\omega-\mathcal{F}'(f_\omega)g_\omega=2\omega f_\omega. \tag{1.30}$$

Since the kernel of L_+ in the radial sector is empty, this equation has a unique solution and it will decay as in (1.26). Consider the difference quotient $\delta f_\omega = (f_{\omega+\delta\omega} - f_\omega)/(\delta\omega)$ which satisfies:

$$L_+\delta f_\omega = 2\omega f_{\omega+\delta\omega} + \delta\omega f_{\omega+\delta\omega} + \int_0^1 \big(\mathcal{F}'\big(f_\omega + \tau(f_{\omega+\delta\omega} - f_\omega)\big) - \mathcal{F}'(f_\omega)\big)\,\mathrm{d}\tau\,\delta f_\omega.$$

Now as noted above $f_{\omega+\delta\omega} \to f_\omega$ in C^4 and the convergence is uniform over all $\mathbb{R}^n$ because of the uniform exponential decay in Lemma 1.5. From this it is clear that $\limsup_{\delta\omega\to 0} \|\delta f_\omega\|_{H^2} < \infty$ and that the right-hand side converges to $2\omega f_\omega$ strongly in L^2 as $\delta\omega \to 0$, and consequently $\delta f_\omega \to g_\omega$ strongly in H^2_{rad}, so that differentiability into H^2 is established. To improve this to H^3, differentiate to get:

$$\begin{aligned}
L_+(\nabla\delta f_\omega) = {} & 2\omega\nabla f_{\omega+\delta\omega} + \delta\omega\nabla f_{\omega+\delta\omega} \\
& + \int_0^1 \big(\mathcal{F}''(f_\tau)\nabla f_\tau - \mathcal{F}''(f_\omega)\nabla f_\omega\big)\,\mathrm{d}\tau\,\delta f_\omega \\
& + \int_0^1 \big(\mathcal{F}'(f_\tau) - \mathcal{F}'(f_\omega)\big)\,\mathrm{d}\tau\,\nabla\delta f_\omega \\
& + \mathcal{F}''(f_\omega)\nabla f_\omega \delta f_\omega,
\end{aligned}$$

where $f_\tau = f_\omega + \tau(f_{\omega+\delta\omega} - f_\omega)$. Most of the argument is exactly as above except that the terms involving $\mathcal{F}''$ have to be treated carefully using condition (E2). To see how this is done consider the term in the second line and rewrite it as:

$$\int_0^1 \left(f_\tau^s \mathcal{F}''(f_\tau)\frac{\nabla f_\tau}{f_\tau^s} - f_\omega^s \mathcal{F}''(f_\omega)\frac{\nabla f_\omega}{f_\omega^s}\right)\mathrm{d}\tau\,\delta f_\omega.$$

Since $s < 1$ Lemma 1.5 together with (E2) implies that $f_\tau^s\mathcal{F}''(f_\tau)$ converges uniformly to $f_\omega^s\mathcal{F}''(f_\omega)$, while $\frac{\nabla f_\tau}{f_\tau^s}$ converges uniformly to $\frac{\nabla f_\omega}{f_\omega^s}$. This reduces the problem to the situation just treated. The arguments that the map is C^2 into H^2 are similar and will be omitted. □

Remark. – In fact the decay estimate (1.26) can be made locally uniform in ω:

LEMMA 1.5. – *Assume $0 \leqslant \omega_0 - \theta < \omega_0 + \theta < m$, then for all $\varepsilon \in (0, \omega_0 - \theta)$ there exists R, M such that*

$$\sup_{|x|\geqslant R,\,\omega\in[\omega_0-\theta,\omega_0+\theta]} \big|f_\omega(x)\big| \leqslant M\mathrm{e}^{-(|\omega_0-\theta|-\varepsilon)|x|}.$$

Proof. – Indeed for $n \geqslant 2$ Strauss' radial lemma ([13,1]) implies that $H^1(\mathbb{R}^n)$ functions are bounded pointwise by $C\|u\|_{H^1}|x|^{-(n-1)/2}$ for $|x| \geqslant r_1$ with C, r_1 depending only n. Also for $n = 1$ even *non-increasing* functions satisfy $u(r) \leqslant r^{-1/2}\|u\|_{L^2}$. The right side of (1.2) defines a C^1 function H^2_{rad} to L^2_{rad} with derivative L_+. The kernel of L_+ is empty in the radial sector and it defines an isomorphism $H^2_{\mathrm{rad}} \to L^2_{\mathrm{rad}}$, so that the implicit function theorem implies the map $\omega \to f_\omega \in H^2_{\mathrm{rad}}$ is continuous. Therefore for all ε there exists r_1 such that

$$\sup_{|x|\geqslant r_1,\,\omega\in[\omega_0-\theta,\omega_0+\theta]} \left|\frac{\mathcal{F}(f_\omega)}{f_\omega}\right| \leqslant \varepsilon.$$

Define

$$M=\sup_{\omega\in[\omega_0-\theta,\omega_0+\theta]} f_\omega(r_1)\mathrm{e}^{mr_1}$$

then (employing the standard technique for proving exponential decay ([6, Section III.7])) the maximum principle gives:

$$f_\omega\big(|x|\big)\leqslant M\mathrm{e}^{-\sqrt{m^2-\omega^2-\varepsilon}|x|}$$

which implies the result. □

Remark. – The hypotheses above are not the most general possible but are sufficiently general to hold for the pure power case with $1<p<1+4/(n-2)$: observe in particular that the regularity hypotheses in (E2) hold since $p>1$.

1.3.2. Formulation of general stability theorem

Before stating the general stability theorem it is important to introduce two further hypotheses. The first, which is elementary for $n=1$, guarantees local *well-posedness in energy norm* for (1.3):

$$\begin{aligned}&\text{(WP) Given } (\phi(0),\psi(0))\in H^1\oplus L^2 \text{ there exists}\\ &T_*=T_*\big(\|(\phi(0),\psi(0))\|_{H^1\oplus L^2}\big) \text{ and a unique weak}\\ &\text{solution to (1.3) with regularity}\\ &(\phi,\psi)\in C\big([0,T_*];H^1\oplus L^2\big)\cap C^1\big([0,T_*];L^2\oplus H^{-1}\big).\\ &\text{Furthermore the quantities } H,Q,\Pi \text{ are preserved}\\ &\text{along the integral curves } t\to\big(\phi(t),\psi(t)\big).\end{aligned}\tag{1.31}$$

Remark. – General conditions on $\mathcal{F}$ which ensure such a local well-posedness result have been given by many authors (see for example [9,7,3] and references therein). In particular in [9, Theorem 2.1 and subsequent remarks] it is proved that (WP) holds for:

(i) the pure power non-linearity with $1<p<1+4/(n-2)$

(ii) a C^1 function $\mathcal{F}$ satisfying $k_1|\mathcal{F}(f)|\leqslant|f\mathcal{F}'(f)|\leqslant k_2|\mathcal{F}(f)|\leqslant k_3|f|^p$ for p as in (i) and for $|f|>k_4>0$ for some positive k_i.

The second hypothesis is necessary for treatment of the non-linear terms in Section 2.4. Introduce the potential $\mathcal{V}(\phi)=G(|\phi|)$ as in (1.4) and make the following hypothesis:

$$\begin{aligned}&\text{(N) If } \mathcal{V}'' \text{ is the second derivative of } \mathcal{V}, \text{ then the map}\\ &\phi\mapsto\mathcal{V}''(\phi) \text{ is continuous as a map } H^1\big(\mathbb{R}^n\big)\to L^a\big(\mathbb{R}^n\big)\\ &\text{for some } a\geqslant\frac{n}{2}.\end{aligned}$$

This condition holds under appropriate Holder or Lipshitz continuity assumptions on $\mathcal{V}''$, e.g. for the pure power case $\mathcal{V}(\phi)=|\phi|^{p+1}/(p+1)$ with $p>1$: see the discussion following Lemma A.2.

THEOREM 1.6 (General stability theorem). – *Let β be such that* (E1)–(E4), (U1)–(U2) (S1) (WP) *and* (N) *hold, and define*:

$$O_{\text{stab}}\equiv\left\{(\omega,\theta,\xi,u)\in O\colon\ \frac{\partial}{\partial\omega}\big(-\omega\|f_\omega\|_{L^2}^2\big)>0\right\}.\tag{1.32}$$

Then for all $\lambda_0 \in O_{\text{stab}}$ *there exists* $\varepsilon_* = \varepsilon_*(\lambda_0) > 0$ *such that if*

$$\varepsilon = \big\|\phi(0,\cdot) - \phi_s(\cdot;\lambda_0)\big\|_{H^1} + \big\|\psi(0,\cdot) - \psi_s(\cdot;\lambda_0)\big\|_{L^2} < \varepsilon_*(\lambda_0)$$

the conclusions of Theorem 1.1 *hold.*

Remark. – The proof of this theorem is given in Section 2, based upon an infinitesimal stability Theorem 2.7 which generalises Theorem 1.3 to general β. The other main input to the theorem is the modulation theory described in Section 1.2.1 and developed fully in Section 2.5.

2. Proof of general stability theorem

In this section the proof of Theorem 1.6 will be given while the derivation of the modulation equations and the proof of the positivity of $\mathcal{E}$ on Υ_λ are postponed to Sections 2.5 and 2.6.

2.1. Strategy of the proof

By assumption (WP) there exists a solution on some interval $[0, T_*]$ with regularity (1.31). The plan is to use the conservation laws (1.4), (1.19) and (1.20) and the decomposition of the solution as:

$$\phi(t,x) = \mathrm{e}^{\mathrm{i}\Theta}\big(f_\omega(Z) + v(t,x)\big), \tag{2.1}$$

$$\psi(t,x) = \mathrm{e}^{\mathrm{i}\Theta}\big(\mathrm{i}\omega\gamma f_\omega(Z) - \gamma u\cdot\nabla_Z f_\omega(Z) + w(t,x)\big), \tag{2.2}$$

where

$$Z = Z(t,x) = \mathbf{Z}\big(x;\lambda(t)\big) = \gamma P_u\big(x - \xi(t)\big) + Q_u\big(x - \xi(t)\big), \tag{2.3}$$

$$\Theta = \Theta(t,x) = \boldsymbol{\Theta}\big(x;\lambda(t)\big) = \theta(t) - \omega u\cdot Z, \tag{2.4}$$

$$\gamma = \overline{\boldsymbol{\gamma}}\big(u(t)\big) = \big(1 - |u(t)|^2\big)^{-1/2}, \tag{2.5}$$

to continue the solution globally with $\|(v,w)\|_{H^1\oplus L^2}$ small. This system is clearly underdetermined as it stands and further conditions must be imposed to determine the function $t \to \lambda(t)$. This will be achieved with the requirement that the pair (v,w) satisfies

$$\mathcal{C}_A\big(v(t), w(t); \lambda(t)\big) = 0 \tag{2.6}$$

for all t and for $A = -1, \dots, 2n+1$, where $\mathcal{C}_A$ are linear functionals defined by:

$$\mathcal{C}_A(v,w;\lambda) = \int_{\mathbb{R}^n} \big(\big\langle w, b_A(Z;\lambda)\big\rangle_{L^2} + \big\langle v, a_A(Z;\lambda)\big\rangle_{L^2}\big)\,\mathrm{d}x, \tag{2.7}$$

where a_A, b_A are the functions defined in (A.44)–(A.51). Thus the following closed subspace

$$\Upsilon_\lambda \equiv \big\{(v,w) \in H^1 \times L^2\colon \mathcal{C}_A(v,w;\lambda) = 0\big\} \tag{2.8}$$

will be of importance. As remarked in Section 1.2.2 the condition $(v,w) \in \Upsilon_\lambda$ is equivalent to the requirement that $(\phi - \phi_S(\lambda), \psi - \psi_S(\lambda)) \in N_\lambda\mathcal{S}$, i.e. that it be symplectically orthogonal to the tangent space. The possibility of writing the *initial data* in this way, with (v,w) satisfying the constraints (2.6), will be proved in Section 2.3 after a discussion of the linearized problem.

2.2. Linearization

The linearization of the equations (1.3) about the solution (1.9)–(1.10) is [3]:

$$\begin{aligned}&\partial_t v + \mathrm{i}\omega\gamma v - w = 0,\\ &\partial_t w + \mathrm{i}\omega\gamma w + M_\lambda v = 0,\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}M_\lambda v = {} & -\Delta_x v + 2\mathrm{i}\omega\gamma u\cdot\nabla_x v + \big(m^2+\omega^2\gamma^2|u|^2\big)v\\ & - f_\omega^{p-1}v - (p-1)f_\omega^{p-1}\Re v,\end{aligned} \tag{2.10}$$

where $f_\omega = f_\omega(Z)$, with Z as in (1.6). Equations (2.9) define a linear operator $\tilde{\mathcal{M}}_\lambda$ in an obvious way:

$$\tilde{\mathcal{M}}_\lambda(v,w) = (-\partial_t v - \mathrm{i}\omega\gamma v + w, -\partial_t w - \mathrm{i}\omega\gamma w - M_\lambda v). \tag{2.9$'$}$$

Restricting this operator to act on functions of Z only leads to the operator $\mathcal{M}_\lambda$ given by:

$$\mathcal{M}_\lambda(v,w) = (\gamma u\cdot\nabla_Z v - \mathrm{i}\omega\gamma v + w, \gamma u\cdot\nabla_Z w - \mathrm{i}\omega\gamma w - M_\lambda v). \tag{2.11}$$

The formal L^2 adjoint $\mathcal{M}_\lambda^*$ is given by:

$$\mathcal{M}_\lambda^*(a,b) = (-\gamma u\cdot\nabla_Z a + \mathrm{i}\omega\gamma a - M_\lambda b, -\gamma u\cdot\nabla_Z b + \mathrm{i}\omega\gamma b + a). \tag{2.12}$$

The following lemma is proved by a straightforward calculation:

LEMMA 2.1. – *If* $\mathcal{M}_\lambda^*(a,b)=0$ *then* $L_+\Re b = 0$, $L_-\Im b = 0$ *and* $a = \gamma u\cdot\nabla_Z b - \mathrm{i}\omega\gamma b$.

Elements of the generalised kernel of $\mathcal{M}_\lambda$ and $\mathcal{M}_\lambda^*$ can be generated by differentiation of (ϕ_S,ψ_S) with respect to the $2n+2$ parameters $\lambda = (\theta,\omega,\xi,u)$ and adjusting for the phase. Thus consider the following set of functions:

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}\Theta}\frac{\partial\phi_S}{\partial\theta} = \mathrm{i}f_\omega,\\ &\mathrm{e}^{-\mathrm{i}\Theta}\frac{\partial\phi_S}{\partial\omega} = g_\omega + \mathrm{i}\left(\frac{t}{\gamma} - u\cdot Z\right)f_\omega,\\ &\mathrm{e}^{-\mathrm{i}\Theta}\frac{\partial\phi_S}{\partial\xi^i} = -\big((\gamma P_u + Q_u)\nabla_Z f_\omega\big)_i - \mathrm{i}\omega\gamma u_i f_\omega,\\ &\mathrm{e}^{-\mathrm{i}\Theta}\frac{\partial\phi_S}{\partial u^i} = -t(\gamma P_u + Q_u)_{ij}\frac{\partial f_\omega}{\partial Z^j} + \zeta_{ji}\frac{\partial f_\omega}{\partial Z_j} + \mathrm{i}\frac{\partial}{\partial u^i}\Theta f_\omega,\end{aligned}$$

where computation gives

$$\frac{\partial\Theta}{\partial u^i}(Z;\lambda) = -\omega\gamma\big((\gamma P_u + Q_u)Z\big)_i, \tag{2.13}$$

$$\frac{\partial Z^j}{\partial u^i}(Z;u) = -t(\gamma P_u + Q_u)_{ij} + \zeta_{ji}(Z;u), \tag{2.14}$$

[3] To be precise the phase is adjusted in this linearisation see (2.1)–(2.2).

with

$$\zeta_{ji}(Z;u) = \gamma^2 (u\cdot Z)(P_u)_{ij} + \frac{(\gamma-1)}{\gamma|u|^2}(u\cdot Z)(Q_u)_{ij} + \frac{(\gamma-1)}{|u|^2}(Q_u Z)_i u_j. \tag{2.15}$$

These functions depend on t as well as ω, u, Z which turns out not to be convenient: it is better to consider linear combinations which are functions of Z only. This can be done at the expense of considering elements of the generalised null space of the linear operators $\mathcal{M}_\lambda, \mathcal{M}_\lambda^*$. The following definitions give appropriate linear combinations which will be used here:

$$(b_0, a_0) = \mathrm{e}^{-\mathrm{i}\Theta}\frac{\partial}{\partial\theta}(\phi_S, -\psi_S), \tag{2.16}$$

$$(b_{-1}, a_{-1}) = \mathrm{e}^{-\mathrm{i}\Theta}\frac{\partial}{\partial\omega}(\phi_S, -\psi_S), \tag{2.17}$$

$$-(\gamma P_u + Q_u)_{ij}(b_j, a_j) = \mathrm{e}^{-\mathrm{i}\Theta}\frac{\partial}{\partial\xi^i}(\phi_S, -\psi_S) - \omega\gamma u^i(b_0, a_0), \tag{2.18}$$

$$(b_{n+i}, a_{n+i}) = \mathrm{e}^{-\mathrm{i}\Theta}\frac{\partial}{\partial u^i}(\phi_S, -\psi_S), \tag{2.19}$$

where ϕ_S, ψ_S are as in (1.9)–(1.10).

LEMMA 2.2. – *The pairs* (2.16)–(2.19), *explicit formulae for which are* $(A.44)$–$(A.51)$ *in the Appendix, lie in the generalised null space of* $\mathcal{M}_\lambda^*$. *To be precise*:

$$\mathcal{M}_\lambda^*(a_i, b_i) = 0, \quad i = 0, \ldots, n, \tag{2.20}$$

while

$$\mathcal{M}_\lambda^*(a_{-1}, b_{-1}) = -\gamma^{-1}(a_0, b_0), \tag{2.21}$$

and

$$\mathcal{M}_\lambda^*(a_{n+i}, b_{n+i}) = \sum_{j=1}^n (\gamma P + Q)_{ij}(a_j, b_j), \quad i = 1, \ldots, n. \tag{2.22}$$

Furthermore the map $\lambda \to (b_A(\cdot;\lambda), a_A(\cdot;\lambda))$ *is continuous from* O *to* $H^3 \times H^2$ *and* C^1 *from* O *to* $H^2 \times H^1$.

Proof. – The first part is proved by direct calculation using the formulae in Appendix A.6. The final regularity statement follows from Theorem 1.4. □

LEMMA 2.3. – *Let* $(b_A, a_A)(Z;\lambda)$ *be defined as in* (2.16)–(2.19). *For* $\lambda \in O_{\mathrm{stab}}$ *the matrix*

$$\overline{\boldsymbol{\Omega}}_{AB} = \langle a_A, b_B\rangle_{L^2} - \langle a_B, b_A\rangle_{L^2}$$

is invertible, i.e., the subspace spanned by the pairs (b_A, a_A) *is symplectic with respect to the structure* $\underline{\boldsymbol{\Omega}}$ *defined in* (1.16).

Proof. – This follows from the fact that the only non-zero matrix elements are (allowing for skew-symmetry):

$$\overline{\boldsymbol{\Omega}}_{-1,0} = -\gamma\frac{\partial}{\partial\omega}\big(\omega\|f_\omega\|_{L^2}^2\big),$$

$$\overline{\boldsymbol{\Omega}}_{n+j,l} = -2\left\langle \gamma u^k\frac{\partial^2 f_\omega}{\partial Z^k\partial Z^j} - \mathrm{i}\omega\gamma\frac{\partial f_\omega}{\partial Z^j}, \zeta_i^l\frac{\partial f_\omega}{\partial Z^l} - \mathrm{i}\omega\gamma(\gamma P_u Z + Q_u Z)^i f_\omega\right\rangle_{L^2}$$

$$+\left\langle\frac{\partial f_\omega}{\partial Z^j},(\gamma P_u+Q_u)_{im}\frac{\partial f_\omega}{\partial Z^m}\right\rangle_{L^2}$$
$$=\gamma^2\left(\frac{\|\nabla f_\omega\|^2_{L^2}}{n}+\omega^2\|f_\omega\|^2_{L^2}\right)(\gamma P_u+Q_u)_{jl},$$

where the summation convention is used. (These formulae are found with use of (A.55), (A.39).) Of the remaining integrals most vanish by inspection but some vanish due to a cancellation depending upon (A.6), (A.55) and (A.39), for example:

$$\overline{\boldsymbol{\Omega}}_{i,-1}=-2\left\langle(\gamma u\cdot\nabla_Z-\mathrm{i}\omega\gamma)\frac{\partial f_\omega}{\partial Z^i},g_\omega-iu\cdot Zf_\omega\right\rangle_{L^2}$$
$$=-\gamma u^i\frac{\langle\nabla f_\omega,\nabla g_\omega\rangle_{L^2}}{n}-\omega\gamma u^i\frac{\|f_\omega\|^2_{L^2}}{2}=0,$$
$$\overline{\boldsymbol{\Omega}}_{0,n+i}=2\omega\gamma\left\langle f_\omega,\zeta_i^k\frac{\partial f_\omega}{\partial Z^k}\right\rangle_{L^2}-2\omega\gamma^2\big\langle u\cdot\nabla_Z f_\omega,(\gamma P_u+Q_u)^i f_\omega\big\rangle_{L^2}$$
$$=\|f_\omega\|^2_{L^2}\left(-\omega\gamma\frac{\partial\zeta_i^k}{\partial Z^k}+\omega\gamma^2u^k(\gamma P_u+Q_u)_{ik}\right)=0.$$

2.3. Preparation of the initial data

The following result, which provides a symplectic normal tubular neighbourhood for $\mathcal{S}$ in the stable case, allows the initial data $(\phi(0),\psi(0))$ to be decomposed as in (2.1)–(2.6).

LEMMA 2.4. – *Assume that for given Cauchy data $(\phi(0),\psi(0))\in H^1\times L^2$ there exists $\lambda_0\in O_{\mathrm{stab}}$ such that*

$$\big\|\phi(0)-\phi_S(\lambda_0)\big\|_{H^1}+\big\|\psi(0)-\psi_S(\lambda_0)\big\|_{L^2}<\delta.$$

Then $\exists c_1>0,\delta_1>0$ such that $\delta<\delta_1$ implies the existence of $\lambda(0)\in O_{\mathrm{stab}}$, depending differentiably upon $(\phi(0),\psi(0))\in H^1\times L^2$, such that $(v(0),w(0))$ defined by:

$$\big(v(0),w(0)\big)=\mathrm{e}^{-\mathrm{i}\Theta(0;\lambda(0))}\big(\phi(0)-\phi_S\big(\lambda(0)\big),\ \psi(0)-\psi_S\big(\lambda(0)\big)\big)\tag{2.23}$$

satisfies

$$\mathcal{C}_A\big(v(0),w(0);\lambda(0)\big)=0\tag{2.24}$$

for $A=-1,\dots,2n+1$, and

$$\big\|\phi(0)-\phi_S\big(\lambda(0)\big)\big\|_{H^1}+\big\|\psi(0)-\psi_S\big(\lambda(0)\big)\big\|_{L^2}<c_1\delta.$$

Also when $(\phi(0),\psi(0))=(\phi_S(\lambda_0),\psi_S(\lambda_0))$ the solution takes the value $\lambda(0)=\lambda_0$.

Proof. – The proof is an application of the implicit function theorem. Rewrite (2.24) as

$$\underline{\boldsymbol{\Omega}}\left(\big(\phi(0)-\phi_S\big(\lambda(0)\big),\ \psi(0)-\psi_S\big(\lambda(0)\big)\big),\ \left(\frac{\partial}{\partial\lambda_A}(\phi_S,\psi_S)\big(\lambda(0)\big)\right)\right)=0.\tag{2.25}$$

When

$$\big(\phi(0),\psi(0)\big)=\big(\phi_S(\lambda_0),\psi_S(\lambda_0)\big)$$

there is a solution $\lambda(0) = \lambda_0$. So local solvability will be a consequence of invertibility of the derivative with respect to λ which can be checked explicitly: referring to (2.16)–(2.19) it can be seen that the derivative is invertible on account of Lemma 2.3. □

The next theorem, which generalises Theorem 1.2, give an interpretation of Lemmas 2.3 and 2.4. Define $T_\lambda\mathcal{S}$ to be the span of $\{\tau_A = \frac{\partial}{\partial\lambda_A}(\phi_S, \psi_S)\}$ and for $\lambda \in O_{\mathrm{stab}}$ let Pr_λ be the projection operator onto this subspace with respect to $\underline{\boldsymbol{\Omega}}$ i.e.

$$\mathrm{Pr}_\lambda\, v = \tilde{\boldsymbol{\Omega}}^{-1}_{AB}\underline{\boldsymbol{\Omega}}(v, \tau_A)\tau_B,$$

where $\tilde{\boldsymbol{\Omega}}_{AB} = \underline{\boldsymbol{\Omega}}(\tau_A, \tau_B)$. (This matrix is related to $\bar{\boldsymbol{\Omega}}_{AB}$ by a simple change of basis as in (2.16)–(2.19) and is non-degenerate for $\lambda \in O_{\mathrm{stab}}$ also.) Let $\mathrm{Qr}_\lambda v = v - \mathrm{Pr}_\lambda v$. Then it follows from consideration of the map

$$T_\lambda\mathcal{S} \oplus N_\lambda\mathcal{S} \to T_{\lambda_0}\mathcal{S} \oplus N_{\lambda_0}\mathcal{S}$$
$$(v_1, v_2) \mapsto (\mathrm{Pr}_{\lambda_0} v_1, \mathrm{Qr}_{\lambda_0} v_2)$$

that $\mathrm{Qr}_{\lambda_0} : N_\lambda\mathcal{S} \to N_{\lambda_0}\mathcal{S}$ is a linear homeomorphism for small $\lambda - \lambda_0$. Define a map:

$$\begin{aligned} &H^1 \oplus L^2 \to \mathbb{R}^{2n+2} \oplus N_{\lambda_0}\mathcal{S} \\ &(\phi, \psi) \mapsto \big(\lambda, \mathrm{Qr}_{\lambda_0}\big(\phi - \phi_S(\lambda), \psi - \psi_S(\lambda)\big)\big) \end{aligned} \tag{2.26}$$

by choosing λ as in Lemma 2.4, in such a way that $(\phi - \phi_S, \psi - \psi_S) \in N_\lambda\mathcal{S}$. (This can be done locally uniquely for (ϕ, ψ) sufficiently close to $\mathcal{S}$.) The map (2.26) is a local C^1 diffeomorphism in a neighbourhood of $(\phi_S(\lambda_0), \psi_S(\lambda_0))$. This shows that restricting to a sufficiently small ball B with centre $\phi_S(\lambda_0), \psi_S(\lambda_0)$ the subset $\mathcal{S}_{\mathrm{stab}} \cap B$ is a submanifold of $X \cap B$ with tangent space $T_\lambda\mathcal{S}$. Furthermore, the restriction of the symplectic form to this subspace is non-degenerate by Lemma 2.3. Thus the following result is proved:

THEOREM 2.5. – *The subset $\mathcal{S}_{\mathrm{stab}} = (\phi_S(\cdot, \lambda), \psi_S(\cdot, \lambda))|_{\lambda\in O_{\mathrm{stab}}} \subset X$ is a local C^1 symplectic submanifold, in particular $\bar{\boldsymbol{\Omega}}$, the restriction of $\underline{\boldsymbol{\Omega}}$, is non-degenerate. As λ approaches the boundary of O_{stab} in O, i.e., as*

$$\frac{\partial}{\partial\omega}\big(-\omega\|f_\omega\|^2_{L^2}\big) \to 0,$$

the restriction of the symplectic form $\bar{\boldsymbol{\Omega}}$ degenerates.

2.4. Completion of the proof

The proof can now be completed with the use of Theorem 2.7 and the following result, to be proved in Section 2.5, which ensures the possibility of writing the solution as in (2.1)–(2.6) on some non-trivial interval $[0, T_1]$.

THEOREM 2.6. – *Let (ϕ, ψ) be a solution to the Cauchy problem for* (1.3) *with regularity as in* (1.31) *satisfying $\|(\phi(t), \psi(t))\|_{H^1\oplus L^2} \leqslant N_0$ on some interval $0 \leqslant t \leqslant T^\sharp$. Let there be given $\lambda_0 = (\omega_0, \theta_0, \xi_0, u_0) \in O_{\mathrm{stab}}$. For positive l introduce the subsets $K_l \subset O_{\mathrm{stab}}$ of the form*:

$$K_l = \big\{\lambda = (\omega, \theta, \xi, u) \in O_{\mathrm{stab}}\colon\ |\omega - \omega_0| + |u - u_0| \leqslant l\big\}, \tag{2.27}$$

and always take l to be small enough so that $K_{2l} \subset O_{\mathrm{stab}}$. Assume further that:

$$\lambda(0) = \big(\omega(0), \theta(0), \xi(0), u(0)\big) \in K_{l/4}$$

and define $(v(0), w(0))$ *by* (2.23) *and assume* (2.24) *holds at* $t = 0$. *Then there exists* $\delta_2 > 0$ *with the property that if*

$$\|v(0)\|_{H^1} + \|w(0)\|_{L^2} = \delta < \delta_2 \quad \text{and} \quad l < \delta_2$$

then there exists $T_1 = T_1(\omega_0, u_0, l, \delta, N_0) \in (0, T^\sharp]$ *and* $\lambda \in C^1([0, T_1]; K_{2l})$ *such that if* $(v(t), w(t))$ *are defined by* (2.1)–(2.2) *then*

$$\mathcal{C}_A\big(v(t), w(t); \lambda(t)\big) = 0 \tag{2.28}$$

for $A \in \{-1, \ldots, 2n+1\}$ *and* $0 \leqslant t \leqslant T_1$. *The curve* $t \to \lambda(t)$ *satisfies the system of ordinary differential equations* (2.46)–(2.49) *and there exists* $c = c(\omega_0, u_0, l, \delta, N_0)$ *such that*:

$$\big|\partial_t \lambda - V(\lambda)\big| \leqslant c\big\|(v, w)\big\|_{H^1 \times L^2}. \tag{2.29}$$

Given (ϕ, ψ) as in (1.31) and $\lambda(0)$ as in Lemma 2.4 let $\lambda \in C^1([0, T_1])$ be determined from Theorem 2.6 so that conditions (2.28) hold. The aim is to use the conservation laws for Q, H, Π, which hold for the class of solutions assumed to exist in the hypothesis (WP) to continue this solution indefinitely. First of all the values of the conserved quantites on the basic solitons (1.9)–(1.10) are as follows:

$$\begin{aligned} H(\phi_S, \psi_S) &= H\Big(\mathrm{e}^{\mathrm{i}\Theta} f_\omega(Z), \mathrm{e}^{\mathrm{i}\Theta}\big(\mathrm{i}\omega\gamma f_\omega(Z) - \gamma u \cdot \nabla_Z f_\omega(Z)\big)\Big) \\ &= h_{\omega,u} = \gamma\left(\frac{\|\nabla f_\omega\|_{L^2}^2}{n} + \omega^2\|f_\omega\|_{L^2}^2\right), \end{aligned} \tag{2.30}$$

$$Q(\phi_S, \psi_S) = q_\omega = -\omega\|f_\omega\|_{L^2}^2, \tag{2.31}$$

$$\begin{aligned} \Pi_i(\phi_S, \psi_S) &= \Pi_i\Big(\mathrm{e}^{\mathrm{i}\Theta} f_\omega(Z), \mathrm{e}^{\mathrm{i}\Theta}\big(\mathrm{i}\omega\gamma f_\omega(Z) - \gamma u \cdot \nabla_Z f_\omega(Z)\big)\Big) \\ &= (p_i)_{w,n} = -\gamma\left(\frac{\|\nabla f_\omega\|_{L^2}^2}{n} + \omega^2\|f_\omega\|_{L^2}^2\right)u^i. \end{aligned} \tag{2.32}$$

The next stage is to expand the conserved quantities defined in Section 1.2.1 using the conditions (2.28) to simplify the resulting expressions. It is expedient to write $\overline{\text{Hess}}\ Q_\lambda(v, w)$ for the term second-order in (v, w) in the expansion of Q at $(\phi_S(\lambda), \psi_S(\lambda))$ and similarly for H, Π; these quantities are just the Hessians of the corresponding functions modified by the phase factor $\mathrm{e}^{\mathrm{i}\Theta}$. The condition $\mathcal{C}_0 = 0$ implies:

$$\begin{aligned} Q(\Phi, \Psi) &= Q\big(f_\omega(Z), \mathrm{i}\omega\gamma f_\omega(Z) - \gamma u \cdot \nabla_Z f_\omega(Z)\big) + \langle \mathrm{i}w, v\rangle_{L^2}, \\ &= -\omega\|f_\omega\|_{L^2}^2 + \langle \mathrm{i}w, v\rangle_{L^2}, \\ &= q_\omega + \overline{\text{Hess}}\ Q_\lambda(v, w), \end{aligned} \tag{2.33}$$

while $\mathcal{C}_i = 0$ implies

$$\begin{aligned} \Pi_i(\Phi, \Psi) &= \Pi_i\Big(\mathrm{e}^{\mathrm{i}\Theta} f_\omega(Z), \mathrm{e}^{\mathrm{i}\Theta}\big(\mathrm{i}\omega\gamma f_\omega(Z) - \gamma u \cdot \nabla_Z f_\omega(Z)\big)\Big) \\ &\quad + \left\langle w, \frac{\partial v}{\partial x^i} - \mathrm{i}\omega\gamma u^i v\right\rangle_{L^2} \\ &= (p_i)_{\omega,u} + \left\langle w, \left(\frac{\partial}{\partial x^i} - \mathrm{i}\omega\gamma u^i\right)v\right\rangle_{L^2} \\ &= (p_i)_{\omega,u} + \overline{\text{Hess}}\ (\Pi_i)_\lambda(v, w). \end{aligned} \tag{2.34}$$

Further the $n+1$ conditions $\mathcal{C}_0=\mathcal{C}_i=0$ together with (1.2) and Lemma A.2 imply that:

$$
\begin{aligned}
H(\Phi,\Psi) &= H\big(\mathrm{e}^{\mathrm{i}\Theta} f_\omega(Z), \mathrm{e}^{\mathrm{i}\Theta}\big(\mathrm{i}\omega\gamma f_\omega(Z)-\gamma u\cdot\nabla_Z f_\omega(Z)\big)\big)\\
&\quad+\frac{1}{2}\bigg(\|w\|_{L^2}^2+\bigg\|\frac{\partial v}{\partial x^i}-\mathrm{i}\omega\gamma u^i v\bigg\|_{L^2}^2\\
&\quad+m^2\|v\|_{L^2}^2-\beta(f_\omega)\|v\|_{L^2}^2-\beta'(f_\omega)f_\omega(\Re v)^2\bigg)+\mathrm{o}\big(\big\|(v,w)\big\|_{H^1\times L^2}^2\big)\\
&= h_{\omega,u}+\frac{1}{2}\bigg(\|w\|_{L^2}^2+\bigg\|\frac{\partial v}{\partial x^i}-\mathrm{i}\omega\gamma u^i v\bigg\|_{L^2}^2\\
&\quad+m^2\|v\|_{L^2}^2-\beta(f_\omega)\|v\|_{L^2}^2-\beta'(f_\omega)f_\omega(\Re v)^2\bigg)+\mathrm{o}\big(\big\|(v,w)\big\|_{H^1\times L^2}^2\big)\\
&= h_{\omega,u}+\overline{\mathrm{Hess}}\ H_\lambda(v,w)+\mathrm{o}\big(\big\|(v,w)\big\|_{H^1\times L^2}^2\big).
\end{aligned}
\tag{2.35}
$$

A straightforward calculation shows that:

$$
\begin{aligned}
\mathcal{E}(v,w;\lambda) &= \frac{1}{2}|w+\gamma u\cdot\nabla_Z v-\mathrm{i}\omega\gamma v|_{L^2(\mathrm{d}Z)}^2+\frac{1}{2}\langle v_1,L_+v_1\rangle_{L^2}+\frac{1}{2}\langle v_2,L_-v_2\rangle_{L^2}\\
&= \overline{\mathrm{Hess}}\ H_\lambda(v,w)+u^i(\overline{\mathrm{Hess}}\ \Pi_i)_\lambda(v,w)+\frac{\omega}{\gamma}\overline{\mathrm{Hess}}\ Q_\lambda(v,w).
\end{aligned}
$$

Theorem 2.7 implies that if $\lambda(t)\in K_{2l}$ and (2.28) holds then

$$
\mathcal{E}\big(v(t),w(t);\lambda(t)\big)\geqslant c_3\big(\big\|v(t)\big\|_{H^1}^2+\big\|w(t)\big\|_{L^2}^2\big)
\tag{2.36}
$$

for some positive number c_3 determined by the set K_{2l}. Now write

$$
\delta\omega=\delta\omega(t)=\omega(t)-\omega(0),\qquad \delta u=\delta u(t)=u(t)-u(0)
$$

and introduce a continuous non-negative function $\Delta(t)$ given by

$$
\Delta(T)^2=\sup_{0\leqslant t\leqslant T}\big(\big\|v(t)\big\|_{H^1}^2+\big\|w(t)\big\|_{L^2}^2+\big|\delta u(t)\big|^2+\big|\delta\omega(t)\big|^2\big).
$$

Notice that $\lim_{T\to 0}\Delta(T)=\mathrm{O}(\varepsilon)$ where ε is as in Theorem 1.6; thus $\Delta(0)$ may be assumed as small as necessary. Now:

$$
\begin{aligned}
&H\big(\Phi(t),\Psi(t)\big)+u^i(0)\Pi_i\big(\Phi(t),\Psi(t)\big)+\frac{\omega(t)}{\gamma(u(0))}Q\big(\Phi(t),\Psi(t)\big)\\
&\quad= h_{\omega,u(0)}+u^i(0)(p_i)_{\omega,u}+\frac{\omega(t)}{\gamma(u(0))}q_\omega+\mathcal{E}\big(v(t),w(t);\lambda(t)\big)+\mathrm{o}\big(\Delta(t)^2\big).
\end{aligned}
$$

Consider the Taylor expansion around $u(0)$; using the identities in (A.18)–(A.22) it is straightforward to calculate that:

$$
\begin{aligned}
h_{\omega,u}+u^i(0)(p_i)_{\omega,u} &= e_{\omega,u(0)}+u^i(0)(p_i)_{\omega,u(0)}\\
&\quad+\frac{1}{2}h_{\omega,u(0)}\big(\gamma\big(u(0)\big)^2P_{u(0)}+Q_{u(0)}\big)_{ij}\delta u^i\delta u^j+\mathrm{o}\big(\Delta^2\big).
\end{aligned}
$$

The conservation laws imply that

$$
\begin{aligned}
&H\big(\phi(t),\psi(t)\big)+u^i(0)\Pi_i\big(\phi(t),\psi(t)\big)+\frac{\omega(t)}{\gamma(u(0))}Q\big(\phi(t),\psi(t)\big)\\
&\quad=H\big(\phi(0),\psi(0)\big)+u^i(0)\Pi_i\big(\phi(0),\psi(0)\big)+\frac{\omega(0)+\delta\omega}{\gamma(u(0))}Q\big(\phi(0),\psi(0)\big).
\end{aligned}
$$

Define:

$$
\tilde h_{\omega,u}=h_{\omega,u}+u^i(p_i)_{\omega,u}+\frac{\omega}{\gamma(u)}q_\omega
$$

using which

$$
\begin{aligned}
&\tilde h_{\omega,u(0)}+\frac12 h_{\omega,u(0)}\big(\gamma\big(u(0)\big)^2P_{u(0)}+Q_{u(0)}\big)_{ij}\delta u^i\delta u^j+\mathcal{E}\big(v(t),w(t);\lambda(t)\big)\\
&\quad=\tilde h_{\omega(0),u(0))}+\frac{\delta\omega}{\gamma(u(0))}q_{\omega(0)}+\mathcal{E}\big(v(0),w(0);\lambda(0)\big)+\mathrm{o}\big(\Delta^2\big).
\end{aligned}
$$

Now either compute directly, or use the remark following (1.20), to deduce that:

$$
\frac{\partial\tilde h_{\omega,u}}{\partial\omega}=\frac{1}{\gamma(u)}q_\omega=-\frac{1}{\gamma(u)}\omega\|f_\omega\|^2_{L^2}.
$$

This function is C^1 since the function $\omega\to f_\omega\in H^1_{\mathrm{rad}}$ is C^1. Therefore the Taylor expansion in ω gives

$$
\begin{aligned}
&\frac12\partial^2_\omega\tilde h_{\omega(0),u(0)}(\delta\omega)^2+\frac12 h_{\omega,u(0)}\big(\gamma\big(u(0)\big)^2P_{u(0)}+Q_{u(0)}\big)_{ij}\delta u^i\delta u^j\\
&\quad+\mathcal{E}\big(v(t),w(t);\lambda(t)\big)=\mathcal{E}\big(v(0),w(0);\lambda(0)\big)+\mathrm{o}\big(\Delta^2\big).
\end{aligned}\tag{2.37}
$$

Notice that the first term is positive by the assumption that $\lambda(0)\in K_{l/4}\subset O_{\mathrm{stab}}$. Clearly there exists $\Delta_*>0$ such that if $\Delta(T)<\Delta_*$ for $0\leqslant T\leqslant T_2$ then:

(a) $\lambda(T)\in K_{l/2}$ (since $\lambda(0)\in K_{l/4}$),

(b) (2.37) implies $\Delta(T)^2\leqslant c_4\mathcal{E}(v(0),w(0);\lambda(0))\leqslant c_5\Delta(0)^2$

for some $c_4,c_5>0$. (The constants Δ_*,c_4,c_5 depend upon the set K_{2l} from (2.36): for given λ_0 assume always l to be less than some l_* then Δ_*,c_4,c_5 depend only on λ_0,l_*.) The result now follows by a standard continuation argument. Recall, as remarked above, that $\lim_{T\to0}\Delta(T)=\mathrm{O}(\varepsilon)$. So letting ε in Theorem 1.6 be small it may be assumed that $c_4\mathcal{E}(v(0),w(0);\lambda(0))<\Delta_*^2/2$ and hence that T_2 may be taken to be T_1 and (a) and (b) hold for $0\leqslant T\leqslant T_1$. Again since $\lambda(T)\in K_{l/2}$ for $0\leqslant T\leqslant T_1$ and $\|(v(T),w(T))\|_{H^1\oplus L^2}$ is bounded independent of time, it is possible to bound $\|(\phi,\psi)\|_{H^1\oplus L^2}\leqslant N_0$ uniformly in time and hence to apply Theorem 2.6 repeatedly. This concludes the proof of Theorem 1.6 since estimates (a) and (b) hold uniformly in time and (1.15) follows from (2.29). □

2.5. Proof of Theorem 2.6 (Modulation equations)

The strategy is first to obtain a system of ODE's for $\lambda(t)$ which ensure that the constraints (2.28) are preserved, and then to prove a local existence theorem for this system. Thus assume temporarily there to be given a C^1 function $t\to\lambda(t)\in O_{\mathrm{stab}}$ and define Z,Θ,v,w as in (2.1)–(2.5) and also introduce $\theta_0(t)$ and $C(t)$ by:

$$\theta(t) = \int_0^t \frac{\omega(s)}{\gamma(u(s))}\,\mathrm{d}s - \theta_0(t), \tag{2.38}$$

$$\xi(t) = \int_0^t u(s)\,\mathrm{d}s + C(t). \tag{2.39}$$

Using the formulae for the derivatives of ϕ, ψ in the Appendix A.5, substitution of (2.1)–(2.2) into (1.3) gives:

$$\begin{aligned} &\partial_t v + \mathrm{i}(\omega\gamma + \mu_0)v = w + j_1(Z, \lambda, \dot{\lambda}), \\ &\partial_t w + \mathrm{i}(\omega\gamma + \mu_0)w = -M_\lambda v + j_2(Z, \lambda, \dot{\lambda}) + \mathcal{N}(f_\omega(Z), v), \end{aligned} \tag{2.40}$$

where $\mu_0 = \mu_0(Z; \lambda, \partial_t\lambda) = \partial_t\Theta - \omega\gamma$, M_λ, j_1, j_2 are defined in (2.10), (A.31) and (A.42)–(A.43) and

$$\mathcal{N}(f_\omega, v) = \beta\big(|f_\omega + v|\big)(f_\omega + v) - \beta(f_\omega)f_\omega - \beta(f_\omega)v - f_\omega\beta'(f_\omega)\Re v. \tag{2.41}$$

In the situation that λ depends upon time the functions a_A, b_A (defined in (A.44)–(A.51)) are no longer in the generalised null space of $\mathcal{M}_\lambda^*$ but instead satisfy:

$$\begin{aligned} &\tilde{\mathcal{M}}_\lambda^*(a_A, b_A) = \big(\tilde{\mathbf{I}}_A^1, \tilde{\mathbf{I}}_A^2\big), \quad A = 0, \ldots, n, \\ &\tilde{\mathcal{M}}_\lambda^*(a_{-1}, b_{-1}) = -\gamma^{-1}(a_0, b_0) + \big(\tilde{\mathbf{I}}_{-1}^1, \tilde{\mathbf{I}}_{-1}^2\big), \\ &\tilde{\mathcal{M}}_\lambda^*(a_{n+i}, b_{n+i}) = (\gamma P_u + Q_u)_{ij}(a_j, b_j) + \big(\tilde{\mathbf{I}}_{n+1}^1, \tilde{\mathbf{I}}_{n+i}^2\big), \quad i = 1, \ldots, n, \end{aligned}$$

where $\tilde{\mathcal{M}}_\lambda^*$ is the $L^2(\mathrm{d}x\,\mathrm{d}t)$ adjoint of $\tilde{\mathcal{M}}_\lambda$ defined in (2.9′). Here sums over $j = 1, \ldots, n$ are implied and in these formulae the functions

$$\tilde{\mathbf{I}}_A^\beta = \tilde{\mathbf{I}}_A^\beta(Z; \lambda) = \big\langle \partial_t\lambda - V(\lambda), D_\lambda(a_A, b_A)\big\rangle + \big\langle \partial_t Z + \gamma u, D_Z(a_A, b_A)\big\rangle, \tag{2.42}$$

defined for $\beta = 1, 2$, $A = -1, \ldots, 2n+1$, are exponentially decaying functions in Z. It follows from the final statement of Lemma 2.2 that the map $\lambda \to (\tilde{\mathbf{I}}^2, \tilde{\mathbf{I}}^1)$ is continuous from O to $H^1 \times L^2$. Indeed the exponential decay rate of the $\tilde{\mathbf{I}}^\beta$, which is uniform on compact subsets of O by Lemma 1.5, implies that this map is in fact continuous into the corresponding weighted Sobolev spaces defined by replacing Lebesgue measure $\mathrm{d}x$ by, for example, $(1 + |x|)\,\mathrm{d}x$ in the definition of the norms: this remark will be used below. It is convenient to introduce the matrix $\{D_{AB}\}_{A,B=-1}^{2n+1}$ which has all entries zero except

$$D_{-1,0} = -\gamma^{-1} \quad \text{and} \quad D_{n+i,j} = (\gamma P_u + Q_u)_{ij}, \tag{2.43}$$

so that the preceding equations can be written in a unified way as

$$\tilde{\mathcal{M}}_\lambda^*(a_A, b_A) = D_{A,B}(a_B, b_B) + \big(\tilde{\mathbf{I}}_A^1, \tilde{\mathbf{I}}_A^2\big). \tag{2.44}$$

Integration by parts leads to the following identity:

$$\mathcal{C}_A\big(v(T), w(T); \lambda(T)\big) = \int_{\mathbb{R}^n} \big(\langle a_A, v\rangle + \langle b_A, w\rangle\big)\,\mathrm{d}x\,\Big|_0^T$$

$$= \int_0^T \int_{\mathbb{R}^n} \Big(\big\langle \tilde{\mathbf{I}}^1_A, v \big\rangle + \big\langle \tilde{\mathbf{I}}^2_A, w \big\rangle + D_{A,B} \big(\langle a_B, v \rangle + \langle b_B, w \rangle \big) + \langle a_A, j_1 \rangle + \langle b_A, j_2 \rangle \Big) \, \mathrm{d}x \, \mathrm{d}t, \tag{2.45}$$

from which it follows that the conditions (2.28) will be preserved by the flow if $\lambda(t)$ is chosen so that

$$\int_{\mathbb{R}^n} \Big(\langle a_A, j_1 \rangle + \langle b_A, j_2 + \mathcal{N} \rangle + \big\langle \tilde{\mathbf{I}}^1_A + \mathrm{i}\mu_0 a_A, v \big\rangle + \big\langle \tilde{\mathbf{I}}^2_A + \mathrm{i}\mu_0 b_A, w \big\rangle \Big) \, \mathrm{d}x = 0.$$

Referring to the integrals computed explicitly in Appendix A.7 this leads to the conclusion that the constraints $\mathcal{C}_A(v, w; \lambda) = 0$ are preserved by the evolution if $(\omega(t), \theta_0(t), C(t), u(t))$ evolves according to the system:

$$\partial_t \big(\omega \| f_\omega \|^2_{L^2} \big) = F_{-1}(v, w, \lambda, \partial_t \lambda), \tag{2.46}$$

$$\frac{\partial}{\partial \omega} \big(\omega \| f_\omega \|^2_{L^2} \big) (\dot{\theta}_0 - \omega \gamma u \cdot \dot{C}) = F_0(v, w, \lambda, \partial_t \lambda), \tag{2.47}$$

$$\partial_t \left(\frac{\| \nabla f_\omega \|^2_{L^2}}{n} \gamma u^k \right) + \omega \| f_\omega \|^2_{L^2} \partial_t \big(\omega \gamma u^k \big) = F_i(v, w, \lambda, \partial_t \lambda), \tag{2.48}$$

$$\left(\frac{\| \nabla f_\omega \|^2_{L^2}}{n} + \omega^2 \| f_\omega \|^2_{L^2} \right) \dot{C}^i = F_{n+i}(v, w, \lambda, \partial_t \lambda), \tag{2.49}$$

where

$$F_{-1}(v, w, \lambda, \partial_t \lambda) = \gamma^{-1} \int_{\mathbb{R}^n} \Big(\big\langle \tilde{\mathbf{I}}^1_{-1} + \mathrm{i}\mu_0 a_{-1}, v \big\rangle + \big\langle \tilde{\mathbf{I}}^2_{-1} + \mathrm{i}\mu_0 b_{-1}, w \big\rangle + \langle b_{-1}, \mathcal{N} \rangle \Big) \, \mathrm{d}Z,$$

$$F_0(v, w, \lambda, \partial_t \lambda) = \gamma^{-1} \int_{\mathbb{R}^n} \Big(\big\langle \tilde{\mathbf{I}}^1_0 + \mathrm{i}\mu_0 a_0, v \big\rangle + \big\langle \tilde{\mathbf{I}}^2_0 + \mathrm{i}\mu_0 b_0, w \big\rangle + \langle b_0, \mathcal{N} \rangle \Big) \, \mathrm{d}Z,$$

$$F_i(v, w, \lambda, \partial_t \lambda) = -\big(P_u + \gamma^{-1} Q_u \big)_{ij} \int_{\mathbb{R}^n} \Big(\big\langle \tilde{\mathbf{I}}^1_j + \mathrm{i}\mu_0 a_i, v \big\rangle + \big\langle \tilde{\mathbf{I}}^2_j + \mathrm{i}\mu_0 b_i, w \big\rangle + \langle b_i, \mathcal{N} \rangle \Big) \, \mathrm{d}Z,$$

$$F_{n+i}(v, w, \lambda, \partial_t \lambda) = -\gamma^{-2} \big(\gamma^{-2} P_u + Q_u \big)_{ij} \int_{\mathbb{R}^n} \Big(\big\langle \tilde{\mathbf{I}}^1_{n+j} + \mathrm{i}\mu_0 a_{n+i}, v \big\rangle + \big\langle \tilde{\mathbf{I}}^2_{n+j} + i\mu_0 b_{n+i}, w \big\rangle + \langle b_{n+i}, \mathcal{N} \rangle \Big) \, \mathrm{d}Z.$$

Recall that μ_0 and the $\tilde{\mathbf{I}}'s$ are affine in $\partial_t \lambda$, (see (A.32)–(A.33)), so this system determines $\partial_t \lambda$ only implicitly at best: smallness asumptions on (v, w) are necessary for well-posedness, as will now be explained. For $T \leqslant T^\sharp$ as in the statement of the theorem it follows from hypothesis (WP) that $\partial_t(\phi, \psi)$ are bounded in $L^2 \oplus H^{-1}$ in terms of N_0. Thus if $\lambda \in K_{2l}$ and δ, l are small it will follow that the pair (v, w) is small in $L^2 \oplus H^{-1}$ on some time interval $0 \leqslant t < T^\flat$ with $T^\flat = T^\flat(N_0) \leqslant T^\sharp$. Since the functions $\tilde{\mathbf{I}}^\beta_A$ are smooth and exponentially decaying, this implies by inspection that the system above can be solved for $\partial_t \lambda$ to define a system of ordinary differential equations $\partial_t \lambda = \tilde{V}(\lambda; \phi(t), \psi(t))$. (Notice that the stability condition ensures non-degeneracy of the second equation.) Furthermore it follows from the remark following (2.42) and the continuity of $t \to (\phi, \psi) \in H^1 \times L^2$ that $\tilde{V}$ is a continuous function of λ, t satisfying $|\tilde{V}| \leqslant c = c(l, \delta, N_0)$ for $\lambda(t) \in K_{2l}$ and $0 \leqslant t < T^\flat$. Thus the local existence theorem for ODE's gives a time-interval $T_1 = T_1(l, \delta, N_0) \leqslant T^\flat \leqslant T^\sharp$ and a function $\lambda \in C^1([0, T_1], K_{2l})$ such that (2.46)–(2.49) and hence (2.28) hold on $[0, T_1]$. From this proof it follows that for as long as:

- λ remains inside the compact subset $K_{2l} \subset O_{\mathrm{stab}}$, and
- (v, w) is small enough to apply the argument above,

that $\partial_t(\omega, \theta_0, C, u) \leqslant c\|(v, w)\|_{H^1 \times L^2}$ and hence that (2.29) holds. This completes the proof of Theorem 2.6. □

2.6. Infinitesimal stability

In this section an infinitesimal, or linear, stability theorem which generalises Theorem 1.3, and was used in Section 2.4, will be stated and proved. Throughout t and $\lambda(t) = (\omega(t), \theta(t), \xi(t), u(t))$ are fixed, so that dependence on t will be suppressed. Thus consider a pair $(v, w) \in H^1 \oplus L^2$ of functions of x; since t is fixed, if

$$Z(x) = \gamma P_u(x - \xi) + Q_u(x - \xi), \quad \gamma = \overline{\boldsymbol{\gamma}}(u) = \left(1 - |u|^2\right)^{-1/2},$$

it is possible, without danger of confusion, to consider (v, w) interchangeably as functions of Z rather than of x; derivatives with respect to Z are related to derivatives with respect to x via the chain rule:

$$\frac{\partial v}{\partial Z^j} = \frac{\partial v}{\partial x^i}\left(\gamma (P_u)_{ji} + (Q_u)_{ji}\right), \tag{2.50}$$

$$\Delta_Z + \gamma^2 (u \cdot \nabla_Z)^2 = \Delta_x, \tag{2.51}$$

where $\Delta_Z = (\partial/\partial Z^i)^2$. Correspondingly, in this section the H^1 and L^2 norms are defined in terms of the variable Z; they are equivalent to those defined with the variable x, uniformly for u lying in subsets of the form $\{|u| \leqslant \theta < 1\}$. Recall that L_+, L_- are the differential operators defined in (1.23)–(1.24).

THEOREM 2.7. – *Assume that* (E1)–(E4), (U1)–(U2) *and* (S) *hold and* $\lambda \in O_{\mathrm{stab}}$, *as defined in* (1.32) (*or equivalently in* (1.13) *for the pure power case*). *Then the quadratic form*:

$$\mathcal{E}(v, w; \lambda) = \frac{1}{2}|w + \gamma u \cdot \nabla_Z v - \mathrm{i}\omega\gamma v|^2_{L^2} + \frac{1}{2}\langle v_1, L_+ v_1\rangle_{L^2} + \frac{1}{2}\langle v_2, L_- v_2\rangle_{L^2}, \tag{2.52}$$

where $v = v_1 + \mathrm{i}v_2$, *is equivalent, uniformly for* λ *in compact subsets of* O_{stab}, *to the norm* $\|(v, w)\|_{H^1 \oplus L^2}$ *on the subspace* Υ_λ *defined by* (2.8).

In the next subsection this is proved for the case of the pure power nonlinearity; the general case is dealt with in the following subsection.

2.6.1. Proof of Theorem 2.7 for pure power nonlinearity

The crucial ingredient in the proof of Theorem 2.7 is the following lemma:

LEMMA 2.8. – *Assume that* $\lambda \in O_{\mathrm{stab}}$ (*see* (1.13)) *and that the pair* $(v, w) \in H^1 \oplus L^2$ *is such that* $\mathcal{C}_0(v, w; \lambda) = 0$, *i.e.*,

$$\left\langle w, \mathrm{i} f_\omega(Z)\right\rangle_{L^2} + \left\langle v, \omega\gamma f_\omega(Z) + \mathrm{i}\gamma u \cdot \nabla_Z f_\omega(Z)\right\rangle_{L^2} = 0. \tag{2.53}$$

Then

$$\mathcal{E}(v,w;\lambda)\geqslant 0,$$

and $\mathcal{E}=0$ *implies* $\langle f_\omega, v_1\rangle_{L^2}=0$.

Proof. – Recall from [16] that for $p\in(1,1+4/(n-2))$ the functions f_ω are minimisers of the Gagliardo–Nirenberg quotient $J^{p,n}(u)$ (see (A.7) and surrounding discussion). The positivity of the second derivative of $J^{p,n}$ at f_ω gives the following inequality, used in [17], and valid for all $\zeta\in H^1(\mathbb{R}^n)$:

$$\begin{aligned}&\langle\zeta, L_+\zeta\rangle+\mu_{\omega,n,p}\left\{\left(1-\frac{1}{q}\right)\langle\Delta_Z f_\omega,\zeta\rangle_{L^2}^2-2\left(m^2-\omega^2\right)\langle\Delta_Z f_\omega,\zeta\rangle_{L^2}\langle f_\omega,\zeta\rangle_{L^2}\right.\\&\quad\left.+\left(1-\frac{1}{r}\right)\left(m^2-\omega^2\right)^2\langle f_\omega,\zeta\rangle_{L^2}^2\right\}\geqslant 0,\end{aligned}\tag{2.54}$$

where q,r are the real numbers defined in (A.8) and

$$\mu_{\omega,n,p}=\frac{2q}{|\nabla f_\omega|_{L^2}^2}.\tag{2.55}$$

Completion of the square now gives:

$$\langle\zeta, L_+\zeta\rangle_{L^2}+\Gamma_{\omega,n,p}\langle f_\omega,\zeta\rangle_{L^2}^2\geqslant 0,\tag{2.56}$$

where

$$\Gamma_{\omega,n,p}=\left(m^2-\omega^2\right)^2\mu_{\omega,n,p}\left[1-\frac{1}{r}-\frac{1}{1-\frac{1}{q}}\right].\tag{2.57}$$

This gives:

$$\mathcal{E}(v,w;\lambda)+\frac{1}{2}\Gamma_{\omega,n,p}\langle f_\omega,\zeta\rangle_{L^2}^2\geqslant\frac{1}{2}|w+\gamma u\cdot\nabla_Z v-\mathrm{i}\omega\gamma v|_{L^2}^2+\frac{1}{2}\langle v_2, L_-v_2\rangle_{L^2}.$$

Rewrite (2.53) as

$$2\omega\gamma\langle v_1, f_\omega\rangle_{L^2}+\langle w+\gamma u\cdot\nabla_Z v-\mathrm{i}\omega\gamma v,\mathrm{i}f_\omega\rangle_{L^2}+2\gamma u\cdot\langle v_2,\nabla_Z f_\omega\rangle_{L^2}=0.$$

Now compute that

$$L_-(u\cdot Zf_\omega)=-2u\cdot\nabla_Z f_\omega$$

so that

$$2\omega\gamma\langle v_1, f_\omega\rangle_{L^2}+\langle w+\gamma u\cdot\nabla_Z v-\mathrm{i}\omega\gamma v,\mathrm{i}f_\omega\rangle_{L^2}-\gamma\left\langle L_-^{1/2}v_2, L_-^{1/2}(u\cdot Zf_\omega)\right\rangle_{L^2}=0.$$

But we have:

$$\begin{aligned}\left\|L_-^{1/2}(u\cdot Zf_\omega)\right\|_{L^2(\mathrm{d}Z)}^2&=\left\langle u\cdot Zf_\omega, L(u\cdot Zf_\omega)\right\rangle_{L^2}\\&=\langle u\cdot Zf_\omega,-2u\cdot\nabla_Z f_\omega\rangle_{L^2}\\&=\left\langle u\cdot Z,-u\cdot\nabla_Z\left(|f_\omega|^2\right)\right\rangle_{L^2}\\&=|u|^2\|f_\omega\|_{L^2(\mathrm{d}Z)}^2.\end{aligned}$$

Therefore, recalling that for $\theta \in (0,1)$ and a, b real

$$\left(\theta a + \sqrt{1-\theta^2}\, b\right)^2 \leqslant a^2 + b^2,$$

it follows that

$$\begin{aligned} 4\omega^2 \langle v_1, f_\omega\rangle_{L^2}^2 &\leqslant \|f_\omega\|_{L^2(\mathrm{d}Z)}^2 \left(|w + \gamma u \cdot \nabla_Z v - \mathrm{i}\omega\gamma v|_{L^2(\mathrm{d}Z)}^2 + \langle v_2, L_- v_2\rangle_{L^2}^2\right) \\ &\leqslant 2\|f_\omega\|_{L^2(\mathrm{d}Z)}^2 \left(\mathcal{E}(v, w; \lambda) + \frac{1}{2}\Gamma_{\omega,n,p}\langle f_\omega, v_1\rangle_{L^2}^2\right) \end{aligned}$$

and hence

$$\left(4\omega^2 - \|f_\omega\|_{L^2(\mathrm{d}Z)}^2 \Gamma_{\omega,n,p}\right)\langle f_\omega, v_1\rangle_{L^2}^2 \leqslant 2\|f_\omega\|_{L^2(\mathrm{d}Z)}^2 \mathcal{E}(v, w; \lambda).$$

The conclusion of the lemma will therefore hold if:

$$\left(4\omega^2 - \|f_\omega\|_{L^2(\mathrm{d}Z)}^2 \Gamma_{\omega,n,p}\right) > 0,$$

i.e., if

$$4\omega^2 - \left(m^2 - \omega^2\right)^2 \frac{2q\|f_\omega\|_{L^2(\mathrm{d}Z)}^2}{\|\nabla f_\omega\|_{L^2(\mathrm{d}Z)}^2}\left[1 - \frac{1}{r} - \frac{1}{1 - \frac{1}{q}}\right] > 0. \tag{2.58}$$

Using the fact (see Lemma A.1) that

$$\frac{2(m^2 - \omega^2)q}{|\nabla f_\omega|_{L^2}^2} = \frac{2r}{|f_\omega|_{L^2}^2}$$

and that $r > 0$ if either $n \leqslant 2$ or $p < 1 + 4/(n-2)$, it is easy to calculate that (2.58) holds if:

$$1 > \frac{\omega^2}{m^2} > \frac{1}{a-1}, \quad \text{where } a = \frac{4}{p-1} - n + 2.$$

Proof of Theorem 2.7 (Pure power case). – Assume to the contrary that there exists sequences, $\lambda_\alpha = (\omega_\alpha, \theta_\alpha, \xi_\alpha, u_\alpha) \in \mathbb{R}^{2n+2}$ and $(v_\alpha, w_\alpha) \in H^1 \times L^2$, indexed by $\alpha \in \{1, 2, \ldots\}$, such that for some fixed $\delta_1, \delta_2, \delta_3$ in $(0,1)$:

(i) $|u_\alpha|^2 \leqslant \delta_1 < 1$,

(ii) $\dfrac{1}{a-1} < \delta_2 \leqslant m^{-2}|\omega_\alpha|^2 \leqslant \delta_3 < 1$,

(iii) $\|v_\alpha\|_{L^2}^2 + \|w_\alpha\|_{L^2}^2 = 1$,

(iv) $\mathcal{C}_A(v_\alpha, w_\alpha; \lambda_\alpha) = 0$, for $A = -1, \ldots, 2n+1$,

(v) $\mathcal{E}(v_\alpha, w_\alpha; \lambda_\alpha) \to 0$.

Translation and phase invariance allows the assumption, without loss of generality that θ_α and $(\xi_i)_\alpha$ are zero for all α. By the previous lemma it is known that $\mathcal{E}(v_\alpha, w_\alpha; \lambda_\alpha) \geqslant 0$ and also that

$$\lim_{\alpha\to\infty}\left\langle v_\alpha, f_\omega\left(Z(x; \lambda_\alpha)\right)\right\rangle_{L^2} = 0.$$

Conditions (iii)–(v) together imply that v_α are bounded in H^1, uniformly in α. Thus there exists a subsequence, which will be relabelled as $\alpha = 1, 2, \ldots$, $\lambda = (\omega, 0, 0, u) \in O_{\text{stab}}$ and $(v, w) \in H^1 \times L^2$ such that:

$$\begin{aligned} u_\alpha &\to u \quad \text{in } \mathbb{R}^3, \\ \omega_\alpha &\to \omega \quad \text{in } \mathbb{R}, \\ v_\alpha &\rightharpoonup v \quad \text{in } H^1 \text{ (weakly)}, \\ w_\alpha &\rightharpoonup w \quad \text{in } L^2 \text{ (weakly)}. \end{aligned}$$

The fact that $\lim_{|Z|\to\infty} f_\omega(Z) = 0$ implies that $v \not\equiv 0$. Indeed if this were so Rellich's lemma would imply that $v_\alpha \to 0$ strongly in $L^2(|Z| \leqslant R)$; but combined with the boundedness and uniform decay of f_ω this means that for all $\varepsilon > 0$ there exists $N(\varepsilon)$ such that $\alpha > N$ implies

$$\left|\int \left|\beta(f_\omega)\right|\left|v_\alpha^2\right|\right| + \left|\int \left|\beta'(f_\omega) f_\omega\right| |v_\alpha|^2\right| < \varepsilon. \tag{2.59}$$

This gives a contradiction between (iii) and (v) immediately, so that $v \not\equiv 0$. Now $f_{\omega_\alpha}(Z(x;\lambda_\alpha))$ converges strongly in L^2 to $f_\omega(Z(x;\lambda))$, as do the spatial derivatives, so that

$$\langle v, f_\omega\rangle_{L^2} = 0 \quad \text{and} \quad \mathcal{C}_A(v, w\lambda) = 0. \tag{2.60}$$

Weak lower semicontinuity of the norm combined with (2.59) implies that $\mathcal{E}(v, w; \lambda) = 0$. Furthermore by (2.60) and [17, Proposition 2.7]

$$\langle L_+ v_1, v_1\rangle_{L^2} \geqslant 0$$

so that the limits must satisfy:

$$\begin{aligned} w &= -\gamma u \cdot \nabla_Z v + \mathrm{i}\omega\gamma v, \\ L_+ v_1 &= 0, \\ L_- v_2 &= 0. \end{aligned}$$

Therefore by (1.28)–(1.29) $v = c_j b_j(Z;\lambda) + c_0 b_0(Z;\lambda)$, where b_A are as in (A.44)–(A.47) and c_0 are constants. From this it follows that $w = -c_0 a_0 - c_j a_j$ and hence

$$\mathcal{C}_A(v, w; \lambda) = c_0 \overline{\boldsymbol{\Omega}}_{A0} + c_j \overline{\boldsymbol{\Omega}}_{Aj}.$$

Choosing $A = -1, n+i$ it follows from the formulae derived in Lemma 2.3 that $c_0 = c_j = 0$ contradicting $v \not\equiv 0$. □

2.6.2. Proof of Theorem 2.7 for general nonlinearity

Let $\mathcal{L}_\lambda$ be the symmetric differential operator corresponding to the quadratic form $\mathcal{E}$ i.e., writing $\eta = (v, w)$ the equation

$$\mathcal{E}(v, w; \lambda) = \frac{1}{2}\langle \eta, \mathcal{L}_\lambda \eta\rangle_{L^2}$$

defines $\mathcal{L}_\lambda$; it extends to a self-adjoint operator on H^1. Comparison with (2.12) shows that:

$$\mathcal{L}_\lambda \begin{pmatrix} v \\ w \end{pmatrix} = \mathcal{M}_\lambda^* \begin{pmatrix} w \\ -v \end{pmatrix}. \tag{2.61}$$

Let $J \in \mathrm{GL}(2, \mathbb{C})$ be defined by:

$$J\eta = J(v, w) = (w, -v),$$

then it follows from (1.24) and (1.28)–(1.29) and Lemma 2.1 that the kernel of $\mathcal{L}_\lambda$ is spanned by $Jn_0, \{Jn_i\}_{i=1}^n$, where

$$n_A = (a_A, b_A).$$

The first stage is the following positivity result, analogous to Lemma 2.8, which is apparently related to results in [5]:

LEMMA 2.9. – *Assume that the function β is such that* (E1)–(E4), (U1)–(U2) *and* (S1) *hold and let O_{stab} be defined by* (1.32). *Then for $\lambda \in O_{\text{stab}}$ and $(v, w) \in H^1 \oplus L^2$*

$$\mathcal{E}(v, w; \lambda) \geqslant 0$$

if $\mathcal{C}_0(v, w; \lambda) = 0$.

Proof. – It follows from assumption (S1) that L_+ has a one-dimensional strictly negative eigenspace; let $\tilde{e}$ be the corresponding eigenvector. Then since $L_- \geqslant 0$ it follows that if $\langle v_1, \tilde{e}\rangle_{L^2} = 0$ then $\mathcal{E}(v, w; \lambda) \geqslant 0$ and so $\mathcal{L}_\lambda$ has exactly one negative eigenvalue, with corresponding eigenvector e. The condition $\mathcal{C}_0 = 0$ can be rewritten as $\langle n_0, \eta\rangle_{L^2} = 0$ and therefore according to [17, Lemma E.1] the result will be proved if:

(i) $\langle e, n_0\rangle_{L^2} \neq 0$,
(ii) $n_0 \in (\mathrm{Ker}\ \mathcal{L}_\lambda)^\perp$,
(iii) $\langle \mathcal{L}_\lambda^{-1} n_0, n_0\rangle_{L^2} \leqslant 0$.

The second of these follows from the calculations in the proof of Lemma 2.3 and the fact that $\mathrm{Ker}\,\mathcal{L}_\lambda$ is spanned by $Jn_0, \{Jn_i\}_{i=1}^n$. The third reduces to the condition on ω in the definition of O_{stab} on account of the fact that

$$\mathcal{L}_\lambda(Jn_{-1}) = -\gamma^{-1} n_0 \tag{2.62}$$

by (2.20)–(2.22). Finally to see that (i) reduces to the same condition on ω argue by contradiction: if $\langle e, n_0\rangle_{L^2} = 0$, then (2.62) implies $\langle e, Jn_{-1}\rangle_{L^2} = 0$, which in turn implies, since $\mathcal{L}_\lambda$ is self-adjoint with one-dimensional negative eigenspace spanned by e, that

$$0 \leqslant \langle Jn_{-1}, \mathcal{L}_\lambda Jn_{-1}\rangle_{L^2} = -\gamma \overline{\boldsymbol{\Omega}}_{-1,0}$$

which is negative for $\lambda \in O_{\text{stab}}$ (see Lemma 2.3), giving the required contradiction.

Proof of Theorem 2.7 (General case). – The first stage is exactly as in the pure power case: this provides sequences $(\lambda_\alpha, v_\alpha, w_\alpha)$ converging in the same way to (λ, v, w) with $v \not\equiv 0$. The argument is now completed slightly differently. By the Lagrange multiplier theorem (v, w) satisfies:

$$\mathcal{L}_\lambda(v, w) = c_A n_A + \tilde{c}(v, w),$$
$$\big\langle (v, w), \mathcal{L}_\lambda(v, w)\big\rangle_{L^2} = 0.$$

Taking the inner product with (v, w) implies $\tilde{c} = 0$ and taking the inner product with Jn_B gives $c_A \overline{\boldsymbol{\Omega}}_{AB} = 0$ which implies $c_A = 0$ by Lemma 2.3. Therefore (v, w) lies in the kernel of $\mathcal{L}_\lambda$ and the argument can be finished as in the pure power case.

Appendices

The next two sections collect various identities relating to the soliton profile functions f_ω, the first for very general nonlinearity and the second for the "pure power" case. In the third Appendix the projection operators are defined and some related formulae given. In the fourth and fifth Appendices some integrals and other formulae used in the text are collected together.

A.1. Some integral identities for the f_ω

Multiplication of the equation by f_ω and $x \cdot \nabla f_\omega$ leads in turn to the identities:

$$\|\nabla f_\omega\|_{L^2}^2 + \left(m^2 - \omega^2\right)\|f_\omega\|_{L^2}^2 - \int \beta(f_\omega) f_\omega^2 = 0, \tag{A.1}$$

$$(2-n)\|\nabla f_\omega\|_{L^2}^2 - n\left(m^2 - \omega^2\right)\|f_\omega\|_{L^2}^2 + n\int G(f_\omega) = 0. \tag{A.2}$$

Now define:

$$g_\omega(x) \equiv \frac{\mathrm{d}}{\mathrm{d}\omega} f_\omega(x), \tag{A.3}$$

then differentiation of (A.1) and (A.2) gives, respectively,

$$\begin{aligned} &2\langle \nabla f_\omega, \nabla g_\omega\rangle_{L^2} + 2\left(m^2 - \omega^2\right)\langle f_\omega, g_\omega\rangle_{L^2} \\ &\quad - 2\omega\|f_\omega\|_{L^2}^2 - \int \left(\beta'(f_\omega) f_\omega^2 g_\omega + 2\beta(f_\omega) f_\omega g_\omega\right) = 0, \end{aligned} \tag{A.4}$$

$$\begin{aligned} &(2-n)\langle \nabla f_\omega, \nabla g_\omega\rangle_{L^2} - n\left(m^2 - \omega^2\right)\langle f_\omega, g_\omega\rangle_{L^2} \\ &\quad + n\omega\|f_\omega\|_{L^2}^2 + n\int \beta(f_\omega) f_\omega g_\omega = 0. \end{aligned} \tag{A.5}$$

Using the fact that

$$-\Delta g_\omega + \left(m^2 - \omega^2\right) g_\omega - 2\omega f_\omega = \beta'(f_\omega) f_\omega g_\omega + \beta(f_\omega) g_\omega,$$

it is straightforward to show that

$$\langle \nabla f_\omega, \nabla g_\omega\rangle_{L^2} = \frac{-n\omega}{2}\|f_\omega\|_{L^2}^2. \tag{A.6}$$

A.2. Further properties of the f_ω (Pure power case)

Assume in this subsection that the nonlinearity is a pure power, i.e., $\beta(|\phi|) = |\phi|^{p-1}$. In this case the function f_ω has an alternative variational characterization as minimiser of the Gagliardo–Nirenberg quotient:

$$J^{p,n}(u) \equiv \frac{\|\nabla u\|_{L^2}^{(p-1)n/2}\|u\|_{L^2}^{2+(p-1)(2-n)/2}}{\|u\|_{L^{p+1}}^{p+1}}. \tag{A.7}$$

This is proved in [16] by the direct variational method for $n > 1$; for $n = 1$ it is a classical result of Nagy.

LEMMA A.1. – *Let* $f_\omega \in H^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ *satisfy* (1.2). *Then*

$$\frac{2(m^2-\omega^2)q}{|\nabla f_\omega|_{L^2}^2} = \frac{2r}{|f_\omega|_{L^2}^2} = \frac{(m^2-\omega^2)(p+1)}{|f_\omega|_{L^{p+1}}^{p+1}},$$

where

$$q = \frac{(p-1)n}{4}, \qquad r = 1 + \frac{(p-1)(2-n)}{4}. \tag{A.8}$$

This is proved by combining identities (A.1) and (A.2) and using the fact that $\beta(f)f = \frac{p+1}{2}G(f)$ in this case. Also notice that

$$f_\omega(x) = \left(m^2-\omega^2\right)^{1/(p-1)} f\left(\sqrt{m^2-\omega^2}\,x\right),$$

so that

$$g_\omega(x) = \frac{\mathrm{d}}{\mathrm{d}\omega} f_\omega(x) = \frac{-2\omega}{(p-1)(1-\omega^2)} f_\omega - \frac{\omega}{1-\omega^2} x \cdot \nabla f_\omega,$$

and

$$\int\limits_{\mathbb{R}^n} f_\omega g_\omega \,\mathrm{d}x = \frac{\omega |f_\omega|_{L^2}^2}{2(m^2-\omega^2)} \left(n - \frac{4}{p-1}\right). \tag{A.9}$$

A.3. Taylor's formula for the potential

In this Appendix an estimate for the remainder in Taylor's formula applied to a potential $\mathcal{V}$ as in (N) is given.

LEMMA A.2. – *Assume* $\phi_S \in H^1(\mathbb{R}^n;\mathbb{C})$ *is given and* $\mathcal{V} \in C^2(\mathbb{C};\mathbb{R})$ *satisfies condition* (N) *in Section* 1.3.2: *for all* $\varepsilon > 0$ *there exists a postive number* δ *such that for all* $\phi \in H^1(\mathbb{R}^n;\mathbb{C})$ *satisfying* $\|\phi - \phi_S\|_{H^1} < \delta$

$$\left\| \mathcal{V}(\phi) - \mathcal{V}(\phi_S) - \left\langle \mathcal{V}'(\phi_S), \phi - \phi_S \right\rangle - \frac{1}{2}\left\langle \mathcal{V}''(\phi_S), \phi-\phi_S, \phi-\phi_S \right\rangle \right\|_{L^1} \leqslant \varepsilon \|\phi - \phi_S\|_{H^1}^2.$$

Proof. – At fixed $x \in \mathbb{R}^n$ Taylors formula gives:

$$\begin{aligned} &\mathcal{V}(\phi) - \mathcal{V}(\phi_S) - \left\langle \mathcal{V}'(\phi_S), \phi - \phi_S \right\rangle - \frac{1}{2}\left\langle \mathcal{V}''(\phi_S), \phi-\phi_S, \phi-\phi_S \right\rangle \\ &\quad = \int_0^1 \int_0^1 \sigma \left\langle \mathcal{V}''(\phi_{\sigma\tau}) - \mathcal{V}''(\phi_S), \phi-\phi_S, \phi-\phi_S \right\rangle \mathrm{d}\sigma\,\mathrm{d}\tau, \end{aligned}$$

where $\phi_{\sigma\tau} = \sigma\tau\phi + (1-\sigma\tau)\phi_S$ and $\sigma,\tau \in [0,1]$. Recalling that the Sobolev exponent $2^* = 2n/(n-2)$ it follows from Holder's inequality that this integral can be estimated in L^1 as required if

$$\left\| \mathcal{V}''(\phi_{\sigma\tau}) - \mathcal{V}''(\phi_S) \right\|_{L^p} \leqslant \varepsilon$$

for some $p \geqslant n/2$, the exponent conjugate to $n/(n-2)$. But this follows from the hypothesis (N) since $\phi_{\sigma\tau} - \phi_S = \sigma\tau(\phi - \phi_S)$. $\square$

In order to make use of this result consider the following condition on $\mathcal{V}$:

$$\left|\mathcal{V}''(a)-\mathcal{V}''(b)\right| \leqslant F_1(a,b)+F_2(a,b) \tag{A.10}$$

(or a sum of such terms), with

$$0 \leqslant F_1(a,b) \leqslant c_1|a-b|^{\gamma_1} \quad \text{where either } c_1=0 \text{ or } \gamma_1 \in \left(0, \min\{1, \tfrac{4}{n-2}\}\right] \tag{A.11}$$

$$0 \leqslant F_2(a,b) \leqslant c_2\left(|a|_2^{\beta}+|b|^{\beta_2}\right)|a-b|^{\gamma_2} \quad \text{where either } c_2=0 \text{ or } \beta_2+\gamma_2 \in \left(0, \tfrac{4}{n-2}\right] \text{ and } \gamma_2 \in (0,1]\,. \tag{A.12}$$

Under these conditions on $\mathcal{V}$ Holder's inequality implies the validity of condition (N). Now turning to the case of the pure power nonlinearities for which

$$\mathcal{V}(\phi)=\mathcal{V}_p(\phi)=\frac{|\phi|^{p+1}}{p+1}$$

observe that if $p>1$ these are C^2 functions for which the second derivatives satisfy:

$$\left|\mathcal{V}''(\phi)-\mathcal{V}''(\psi)\right| \leqslant c|\phi-\psi|^{\delta}\left(|\phi|^{p-1-\delta}+|\psi|^{p-1-\delta}\right) \tag{A.13}$$

with $0<\delta<p-1$ if $p<2$ and $\delta=1$ if $p \geqslant 2$.

A.4. Projection operators

The projection operators, along and perpendicular to u, are given by:

$$(P_u)_{ij}=\frac{u^i u^j}{|u|^2}, \qquad (Q_u)_{ij}=\frac{|u|^2\delta_{ij}-u^i u^j}{|u|^2}. \tag{A.14}$$

From this it is straightforward to calculate that

$$\frac{\partial P_{ab}}{\partial u^i}=\frac{u^b\delta_{ia}}{|u|^2}+\frac{u^a\delta_{ib}}{|u|^2}-2\frac{u^a u^b u^i}{|u|^4}, \tag{A.15}$$

$$\frac{\partial P_{ab}}{\partial u^i}P_{bc}=\frac{u^c Q_{ia}}{|u|^2}, \tag{A.16}$$

$$\frac{\partial P_{ab}}{\partial u^i}Q_{bc}=\frac{u^a Q_{ic}}{|u|^2}, \tag{A.17}$$

where the suffix u is temporarily omitted for clarity. Other useful formulae are:

$$\frac{\partial}{\partial u^k}\left(\overline{\boldsymbol{\gamma}}(u)\right)=\overline{\boldsymbol{\gamma}}(u)u^k, \tag{A.18}$$

$$\frac{\partial^2}{\partial u^k\partial u^l}\left(\overline{\boldsymbol{\gamma}}(u)\right)=\overline{\boldsymbol{\gamma}}(u)^5\left(\left(1-|u|^2\right)\delta_{kl}+3u^k u^l\right), \tag{A.19}$$

$$\frac{\partial}{\partial u^k}\left(\overline{\boldsymbol{\gamma}}(u)u^i\right)=\overline{\boldsymbol{\gamma}}(u)^3\left(\left(1-|u|^2\right)\delta_{il}+u^i u^l\right), \tag{A.20}$$

$$\frac{\partial^2}{\partial u^k\partial u^l}\left(\overline{\boldsymbol{\gamma}}(u)u^i\right)=\overline{\boldsymbol{\gamma}}(u)^5\left(\left(1-|u|^2\right)\left(u^k\delta_{il}+u^l\delta_{ik}+u^i\delta_{lk}\right)+3u^i u^k u^l\right), \tag{A.21}$$

$$u^i \frac{\partial^2}{\partial u^k \partial u^l}\big(\overline{\boldsymbol{\gamma}}(u)u^i\big) = \overline{\boldsymbol{\gamma}}(u)^5\big(\big(1-|u|^2\big)|u|^2\delta_{lk} + \big(2+|u|^2\big)u^k u^l\big). \tag{A.22}$$

A.5. Some formulae

In this section some formulae used in the main text are given. Let Z, Θ be the functions defined in (2.3)–(2.5) with the suffix λ temporarily supressed. *It is assumed throughout that whenever either f_ω or g_ω appears it is with argument Z. Also since the time t is fixed it will not be explicitly written.* To start with:

$$\begin{aligned} \frac{\partial Z^j}{\partial x^i} &= \gamma(P_u)_{ji} + (Q_u)_{ji}, \\ \frac{\partial \Theta}{\partial x^i} &= -\omega\gamma u^i. \end{aligned} \tag{A.23}$$

Let ϕ, ψ be as in (2.1)–(2.2) then

$$\frac{\partial \phi}{\partial x^i} = \mathrm{e}^{\mathrm{i}\Theta}\left(\frac{\partial f_\omega}{\partial Z^j}\big(\gamma(P_u)_{ji} + (Q_U)_{ji}\big) + \frac{\partial v}{\partial x^i} - \mathrm{i}\omega\gamma u^i(f_\omega + v)\right), \tag{A.24}$$

$$\begin{aligned} \Delta\phi = \mathrm{e}^{\mathrm{i}\Theta}\bigg(&\frac{\partial^2 f_\omega}{\partial Z^j \partial Z^k}\big(\gamma^2(P_u)_{ji} + (Q_U)_{ji}\big) + \Delta v - \omega^2\gamma^2|u|^2(f_\omega + v) \\ &- 2\mathrm{i}\omega\gamma u\cdot\nabla_x v - 2\mathrm{i}\omega\gamma^2 u\cdot\nabla_Z f_\omega\bigg), \end{aligned} \tag{A.25}$$

$$\frac{\partial \phi}{\partial t} = \mathrm{e}^{\mathrm{i}\Theta}\left(\frac{\partial f_\omega}{\partial Z^j}\frac{\partial Z^j}{\partial t} + \frac{\partial v}{\partial t} + \mathrm{i}\frac{\partial \Theta}{\partial t}(f_\omega + v) + \partial_t\omega g_\omega\right), \tag{A.26}$$

$$\begin{aligned} \frac{\partial \psi}{\partial t} = \mathrm{e}^{\mathrm{i}\Theta}\bigg(&\mathrm{i}\omega\gamma\frac{\partial f_\omega}{\partial Z^j}\frac{\partial Z^j}{\partial t} - \gamma u^k\frac{\partial^2 f_\omega}{\partial Z^j\partial Z^k}\frac{\partial Z^j}{\partial t} + \frac{\partial w}{\partial t} \\ &+ \mathrm{i}\frac{\partial\Theta}{\partial t}(\mathrm{i}\gamma\omega f_\omega - \gamma u\cdot\nabla_Z f_\omega + w) \\ &+ \mathrm{i}\partial_t(\omega\gamma)f_\omega - \partial_t(\gamma u)\cdot\nabla_Z f_\omega \\ &+ (\mathrm{i}\omega\gamma g_\omega - \gamma u\cdot\nabla_Z g_\omega)\partial_t\omega\bigg). \end{aligned} \tag{A.27}$$

To compute time derivatives explicitly, notice first of all that

$$\left.\frac{\partial Z^j}{\partial u^i}\right|_\xi (Z;u) = -t(\gamma P_u + Q_u)_{ij} + \zeta_{ji}(Z;u), \tag{A.28}$$

while

$$\left.\frac{\partial \Theta}{\partial u^i}\right|_\theta (Z;\omega,u) = -\omega Z^i - \omega u^j\zeta_{ji}(Z;u), \tag{A.29}$$

where $\zeta_{ji}(Z;u)$ was defined in (2.15). Now define:

$$\begin{aligned} \mu_j(Z;\lambda,\dot\lambda) &\equiv \partial_t Z^j + \gamma u^j \\ &= \zeta_{jk}(Z)\partial_t u^k - \big((\gamma P_u + Q_u)(\partial_t C)\big)^j, \end{aligned} \tag{A.30}$$

so that the identities

$$u^j \zeta_{jk}(Z;u) = \gamma^2 u \cdot Z u_k + (\gamma-1)(Q_u Z)_k,$$
$$u^j \zeta_{jk}(Z;u) + Z^k = \gamma\big((\gamma P_u + Q_u)Z\big)^k,$$

imply

$$\begin{aligned}\mu_0(Z;\lambda,\dot\lambda) &\equiv \partial_t \Theta - \omega\gamma \\ &= -\partial_t\theta_0 - \partial_t\omega(u\cdot Z) - \omega\gamma\,\partial_t u_k\big(\gamma(P_u Z)_k + (Q_u Z)_k\big) \\ &\quad + \omega\gamma u\cdot\partial_t C.\end{aligned} \tag{A.31}$$

Since the functions μ are all affine in Z it is convenient to decompose them, in the obvious way: as

$$\mu_0 = \tilde\mu_0 + \mu_a^0 Z^a, \qquad \tilde\mu_0 = -\dot\theta_0 + \omega\gamma u\cdot\dot C, \tag{A.32}$$

$$\mu_j = \tilde\mu_j + \mu_a^j Z^a, \qquad \tilde\mu_j = -(\gamma P_u + Q_u)_{jk}\dot C_k. \tag{A.33}$$

LEMMA A.3. – *The μ^j and μ^0 satisfy*

$$\mu_a^0 = \frac{\partial\mu^0}{\partial Z^a} = \big(\gamma^{-1}P_u + Q_u\big)_{ij}(-\omega\gamma u_j)_t, \tag{A.34}$$

$$\mu_a^a = \frac{\partial\mu^i}{\partial Z^i} = \gamma^{-1}\dot\gamma, \tag{A.35}$$

$$\begin{aligned}u^k\mu_k^j &= u\cdot\nabla_Z\mu^j = u\cdot\nabla_Z\frac{\partial Z^j}{\partial t} \\ &= \big(\gamma^2|u|^2(P_u)_{jk} + \big(1-\gamma^{-1}\big)(Q_u)_{jk}\big)\dot u_k,\end{aligned} \tag{A.36}$$

$$\begin{aligned}u^l\mu_a^l &= \frac{\partial}{\partial Z^a}\big(u^l\mu^l\big) = \frac{\partial}{\partial Z^a}\big(u^l\partial_t Z^l\big) \\ &= (\gamma-1)\big(Q_u + \gamma^2 P_u\big)_{aj}\dot u^j.\end{aligned} \tag{A.37}$$

And consequently, if

$$\zeta_{ia}^j = \frac{\partial\zeta_i^j}{\partial Z^a} \tag{A.38}$$

then

$$\zeta_{il}^l = \gamma^2 u_i, \tag{A.39}$$

$$u^a\zeta_{ia}^j = \gamma^2|u|^2(P_u)_{ij} + \frac{\gamma-1}{\gamma}(Q_u)_{ij}, \tag{A.40}$$

$$u^l\zeta_{ij}^l = \gamma^2 u_j u_i + (\gamma-1)(Q_u)_{ij}. \tag{A.41}$$

Proof. – The first identity is obtained by writing

$$\begin{aligned}\frac{\partial\mu^0}{\partial Z^i} &= \big(\gamma^{-1}P + Q\big)_{ij}\frac{\partial\mu^0}{\partial x^j} \\ &= \big(\gamma^{-1}P + Q\big)_{ij}\frac{\partial}{\partial t}\frac{\partial\Theta}{\partial x^j} \\ &= -\big(\gamma^{-1}P + Q\big)_{ij}\partial_t\big(\omega\gamma u^j\big).\end{aligned}$$

The second is proved from the change of variables formula as follows: consider an arbitrary smooth time-independent function $\Gamma(Z)$ and let $G(t,x) = \Gamma(Z(t,x))$. The Jacobian of the transformation $x \mapsto Z$ is γ, so that the integral

$$\gamma \int G(t,x)\,\mathrm{d}x$$

is independent of time; differentiation with respect to t and integration by parts gives the result. To get the third write $u\cdot\nabla_Z = \gamma^{-1}u\cdot\nabla_x$ and computing the derivatives directly. The fourth is obtained by writing $u^l\partial_t Z^l = \partial_t(u\cdot Z) - \dot{u}\cdot Z$ and then expressing $u\cdot Z = \gamma u\cdot(x-\xi)$ and writing the Z derivatives in terms of x derivatives which can be interchanged with the t derivative as in the derivation of the first identity. □

Using the definitions of μ_0, μ_j in (A.24)–(A.27) and substituting into (1.3) leads to (2.40) with error terms given by:

$$j_1(Z;\lambda,\dot{\lambda}) = -\partial_t\omega g_\omega - \mu_j\frac{\partial f_\omega}{\partial Z_j} - \mathrm{i}\mu_0 f_\omega, \tag{A.42}$$

$$\begin{aligned} j_2(Z;\lambda,\dot{\lambda},v) &= (\gamma u)_t\cdot\nabla_Z f_\omega - \mathrm{i}(\omega\gamma)_t f_\omega - \partial_t\omega(\mathrm{i}\omega\gamma g_\omega - \gamma u\cdot\nabla_Z g_\omega) \\ &\quad - \mu_j(\mathrm{i}\omega\gamma - \gamma u\cdot\nabla_Z)\frac{\partial f_\omega}{\partial Z_j} - \mathrm{i}\mu_0(\mathrm{i}\omega\gamma - \gamma u\cdot\nabla_Z) f_\omega. \end{aligned} \tag{A.43}$$

A.6. The generalised null space

Direct calculation, using some identities in Section A.5, shows that the functions in (2.16)–(2.19) are given by:

$$b_0(Z;\lambda) = \mathrm{i} f_\omega(Z), \tag{A.44}$$

$$b_{-1}(Z;\lambda) = g_\omega - \mathrm{i}u\cdot Z f_\omega, \tag{A.45}$$

$$b_i(Z;\lambda) = \frac{\partial f_\omega}{\partial Z_i}, \tag{A.46}$$

$$b_{n+i}(Z;\lambda) = \zeta_{ji}\frac{\partial f_\omega}{\partial Z_j} - \mathrm{i}\omega\gamma\big(\gamma(P_u Z)_i + (Q_u Z)_i\big) f_\omega, \tag{A.47}$$

for $i = 1,\ldots,n$, and

$$a_0(Z;\lambda) = (\gamma u\cdot\nabla_Z - \mathrm{i}\omega\gamma)b_0, \tag{A.48}$$

$$a_{-1}(Z;\lambda) = -\gamma^{-1}b_0 + (\gamma u\cdot\nabla_Z - \mathrm{i}\omega\gamma)b_{-1}, \tag{A.49}$$

$$a_i(Z;\lambda) = (\gamma u\cdot\nabla_Z - \mathrm{i}\omega\gamma)b_i, \quad i = 1,\ldots,n, \tag{A.50}$$

$$a_{n+i}(Z;\lambda) = (\gamma P_u + Q_u)_{ij}b_j + (\gamma u\cdot\nabla_Z - \mathrm{i}\omega\gamma)b_{n+i}, \quad i = 1,\ldots,n. \tag{A.51}$$

A.7. Some integrals

Let $S^{n-1}\subset\mathbb{R}^n$ be the unit sphere and $|S^{n-1}|$ its volume. Then if $\{Z_i\}_{i=1}^n$ are standard co-ordinates on $\mathbb{R}^n$ then:

$$\left|S^{n-1}\right|^{-1}\int\limits_{S^{n-1}} Z_i Z_j Z_k Z_l = \frac{\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{n(n+2)}. \tag{A.52}$$

From this follows directly

$$\int_{\mathbb{R}^n} \frac{\partial^2 f_\omega}{\partial Z^i \partial Z^j} \frac{\partial f_\omega}{\partial Z^k} Z^l \,\mathrm{d}Z = \frac{\|\nabla f_\omega\|_{L^2}^2}{2n} (\delta_{ij}\delta_{lk} - \delta_{li}\delta_{kj} - \delta_{lj}\delta_{ki}), \tag{A.53}$$

and

$$\int_{\mathbb{R}^n} \left(\frac{\partial^2 f_\omega}{\partial Z^i \partial Z^j} \frac{\partial f_\omega}{\partial Z^k} Z^l - \frac{\partial^2 f_\omega}{\partial Z^k \partial Z^j} \frac{\partial f_\omega}{\partial Z^i} Z^l \right) \mathrm{d}Z = \frac{\|\nabla f_\omega\|_{L^2}^2}{n} (\delta_{ij}\delta_{lk} - \delta_{li}\delta_{kj}), \tag{A.54}$$

$$u^k \int_{\mathbb{R}^n} \frac{\partial^2 f_\omega}{\partial Z^k \partial Z^j} \zeta_i^l \frac{\partial f_\omega}{\partial Z^l} \,\mathrm{d}Z = -\frac{\gamma |u|^2}{2n} \|\nabla f_\omega\|_{L^2}^2 (\gamma P_u + Q_u)_{ij}. \tag{A.55}$$

Let b_A, a_A be the pairs of functions defined in (A.44)–(A.51). Then, writing L^2 for $L^2(\mathrm{d}Z)$,

$$\begin{aligned}
\langle b_0, j_2\rangle_{L^2} + \langle a_0, j_1\rangle_{L^2} &= -\partial_t(\omega\gamma)\|f_\omega\|_{L^2}^2 - 2\omega\gamma\,\partial_t\omega\langle f_\omega, g_\omega\rangle_{L^2} \\
&\quad + \omega\gamma\|f_\omega\|_{L^2}^2 \frac{\partial \mu^j}{\partial Z^j} - \frac{1}{2}\|f_\omega\|_{L^2}^2 \gamma u^j \frac{\partial \mu^0}{\partial Z^j} + \frac{1}{2}\|f_\omega\|_{L^2}^2 \gamma u^j \frac{\partial \mu^0}{\partial Z^j} \\
&= -\partial_t\big(\omega\gamma\|f_\omega\|_{L^2}^2\big) + \omega\gamma\|f_\omega\|_{L^2}^2 \frac{\partial \mu^j}{\partial Z^j} \\
&= -\gamma\,\partial_t\big(\omega\|f_\omega\|_{L^2}^2\big), \\
\langle b_{-1}, j_2\rangle_{L^2} + \langle a_{-1}, j_1\rangle_{L^2} &= \gamma^{-1}\tilde{\mu}^0\|f_\omega\|_{L^2}^2 \\
&\quad + u^j\tilde{\mu}^j \left(\frac{-2\gamma}{n} \langle \nabla g_\omega, \nabla f_\omega\rangle_{L^2} - \omega\gamma\|f_\omega\|_{L^2}^2 \right) \\
&\quad + \gamma\tilde{\mu}^0\big(2\omega\langle f_\omega, g_\omega\rangle_{L^2} + |u|^2\|f_\omega\|_{L^2}^2\big) \\
&= \big(\gamma^{-1} + \gamma|u|^2\big)\|f_\omega\|_{L^2}^2\tilde{\mu}^0 + 2\omega\gamma\tilde{\mu}^0\langle f_\omega, g_\omega\rangle_{L^2} \\
&= \gamma\frac{\partial}{\partial\omega}\big(\omega\|f_\omega\|_{L^2}^2\big)\tilde{\mu}^0, \\
\langle b_i, j_2\rangle_{L^2} + \langle a_i, j_1\rangle_{L^2} &= \frac{1}{n}\|\nabla f_\omega\|_{L^2}^2 \partial_t(\gamma u^i) + 2\dot{\omega}(\gamma u^j)\left\langle \frac{\partial g_\omega}{\partial Z^j}, \frac{\partial f_\omega}{\partial Z^i} \right\rangle_{L^2} \\
&\quad + \gamma u^k \int \mu^j \left(\frac{\partial^2 f_\omega}{\partial Z^j \partial Z^k} \frac{\partial f_\omega}{\partial Z^i} - \frac{\partial^2 f_\omega}{\partial Z^i \partial Z^k} \frac{\partial f_\omega}{\partial Z^j} \right) \mathrm{d}Z \\
&\quad + 2\omega\gamma\left\langle \mu^0 f_\omega, \frac{\partial f_\omega}{\partial Z^i} \right\rangle_{L^2} \\
&= (P_u + \gamma Q_u)_{ik}\left\{ \partial_t\left(\frac{\|\nabla f_\omega\|_{L^2}^2}{n}\gamma u^k \right) + \omega\|f_\omega\|_{L^2}^2 \partial_t\big(\omega\gamma u^k\big) \right\}, \\
\langle b_{n+i}, j_2\rangle_{L^2} + \langle a_{n+i}, j_1\rangle_{L^2} &= -(\gamma P + Q)_{ij}\tilde{\mu}^j \left\langle \frac{\partial f_\omega}{\partial Z^k}, \frac{\partial f_\omega}{\partial Z^j} \right\rangle_{L^2} \\
&\quad + 2\gamma u^k\tilde{\mu}^j \left\langle \frac{\partial^2 f_\omega}{\partial Z^j \partial Z^k}, \zeta_i^l \frac{\partial f_\omega}{\partial Z^l} \right\rangle_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+2\omega^2\gamma^2\tilde{\mu}^j\left\langle(\gamma P_u Z+Q_u Z)^i f_\omega,\frac{\partial f_\omega}{\partial Z^j}\right\rangle_{L^2}\\
&+2\omega\gamma\tilde{\mu}^0\left\langle\zeta_i^l f_\omega,\frac{\partial f_\omega}{\partial Z^l}\right\rangle_{L^2}\\
&-2\omega\gamma^2u^k\tilde{\mu}^0\left\langle\frac{\partial f_\omega}{\partial Z^k},(\gamma P_u Z+Q_u Z)^i f_\omega\right\rangle_{L^2}\\
={}&\left(\frac{\|\nabla f_\omega\|_{L^2}^2}{n}+\omega^2\|f_\omega\|_{L^2}^2\right)\gamma^2\big(\gamma^2P_u+Q_u\big)_{ij}\dot{C}^j.
\end{aligned}
$$

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